Actuarial Par Spread and Empirical Pricing of CDS by Decomposition

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INTRODUCTION

A credit default swap (CDS) is a financial contract that works much like typical insurance. The contract defines a reference instrument (a bond) issued by some reference entity (the obligor). The buyer of a CDS pays periodic premiums in exchange for protection against potential losses incurred on the reference instrument when the obligor defaults. Buyers may or may not hold the reference instrument because CDS are often used as a cross hedge for credit exposures arising from other business dealings with the obligor; for example, a supply chain relationship. Of course, CDS can also be used for speculative purposes just like other financial instruments. CDS were invented in the 1990s and steadily gained popularity to reach a peak in 2007 with over 60 trillion USD notional outstanding. The onset of the global financial crisis in 2008 put a dent in its popularity, and CDS were blamed as a main contributor to the global financial crisis. Even so, CDS are still a multi-trillion market these days because they play a vital risk management function.

In earlier years of CDS trading, the spread was generally set without involving an upfront fee. We will refer to such a spread as CDS par spread. Since the introduction of the CDS Big Bang Protocol in April 2009, CDS have been traded with a fixed coupon at either 100 or 500 basis points with an upfront fee to offset the mispricing caused by the fixed coupon rate. Obviously, this change makes secondary market trading of CDS easier, because the upfront fee can freely move with market conditions and clear offsetting trades without having to pile new CDS on existing CDS. A fixed-coupon CDS with an upfront fee can be easily quoted as a CDS par spread. In this paper, we will stick to the CDS par spread in expressing the price of a CDS much like using effective yield instead of coupon rate to quote the cost/return of a bond.

When probabilities of default (PDs) exhibit a term structure behavior, CDS spreads become some complex aggregate of the PDs over the segment of the term structure defined by the tenor of CDS. For a 5-year CDS, one should note that the CDS spread may not be closely aligned with the 5-year PD (annualized) because CDS premiums are only payable up to the time of default. Since the maximum loss given default is 100% and the highest possible PD is also 100%, one would think that the CDS spread should be capped at 10,000 basis points. But Greek sovereign CDS were actually traded at a spread beyond 20,000 basis points prior to its default in the early part of 2012. This is because with Greece's imminent default, the protection buyer was only expected to pay a fraction of the premium determined by a short accrual period, but the protection seller was expected to soon incur a big loss whose amount was not tied to the accrual period. Pushing up the CDS premium beyond 20,000 basis points was to offset the likely short accrual period.

Instead of focusing on PDs over various periods extending out from the point of prediction, one can consider forward PDs (a fixed duration, say, one month, but with different forward starting times). Forward PDs work much like forward interest rates. Due to their contractual features, CDS spreads are expected to be more in line with short-term (long-term) forward PD when shorter-term forward PDs are higher (lower) than longer-term forward PDs. This is because an imminent default suggests high shorter-term forward PDs being coupled with lower longer-term forward PDs. As in the preceding discussion on the Greek default, CDS spreads will have to be high enough to reflect a shorter accrual period due to the imminent default.

The complexity of having to use the term structure of forward PDs to analyze CDS spreads highlights the need for a simpler summary measure. Indeed, that is precisely the objective of this paper. We envisage a CDS premium rate that would have been charged if market participants were risk-neutral and no money changed hands initially. In short, it is equivalent to pricing CDS purely based on their actuarial values. Such a premium rate is referred to as an actuarial par spread and can be computed with availability of the term structure of physical PDs. We show by a numerical example the mechanics of its computation and

explain how it can be operationalized on a daily and live basis by leveraging the corporate default prediction system developed and maintained by the Risk Management Institute (RMI) of the National University of Singapore.

As an example to demonstrate this new measure, this paper will look at Eastman Kodak, which filed for Chapter 11 bankruptcy protection on January 19, 2012. Kodak later emerged from bankruptcy on September 3, 2013, and its stocks under a different ticker began to trade in NYSE on November 1, 2013. We use the actuarial par spread to study the CDS spreads of Kodak over the one-year period leading to the day of its bankruptcy filing. The results suggest that the log-ratio of CDS spread over its corresponding actuarial par spread is highly predictable with its lagged value, and this predictive relationship can form a good basis for empirical pricing of CDS.

I. TERM STRUCTURE OF PHYSICAL PDs AND ACTUARIAL PAR SPREADS

A CDS contract comprises the premium and protection legs. The protection buyer typically pays premiums on a quarterly basis to the seller, which are calculated based on the fixed spread until the reference entity defaults. In exchange, the protection buyer receives a contingent lump sum payment at the default time of the reference entity, and the settlement amount is by convention based on the recovery rate on the reference instrument determined at the credit event auction held specifically for the defaulted obligor within one month of default. Consider at time t a CDS contract to be terminated at time $T > t$. The premium leg comes with periodic future payments at $\{t_1, t_2, \ldots, t_k\}$ with $t_k = T$. The payment date at or immediately before t is denoted by t_0. The protection leg pays an amount set according to one minus a random recovery rate, R_τ at the random default time $\tau > t$ if default occurs before T, at which time all scheduled premium payments beyond the default time cease to apply.

CDS premiums are subject to day count conventions similar to coupon bonds. Adding to the

complexity is the fact that accrual periods do not always end on payment dates. Differences are typically one day, and may be ignored for many practical purposes. For a more precise modeling of CDS, however, we introduce a set of accrual period end dates, $\{t'_1, t'_2, \ldots, t'_k\}$. Let $A(t_{i-1}, t'_i)$ stand for the length of an accrual period measured as the fraction of a year using the appropriate day count convention. For a typical CDS and if a particular quarterly accrual period from t_{i-1} to t'_i inclusive has, say, 91 calendar days, the actual/360 day count convention will give rise to $A(t_{i-1}, t'_i) = 91/360$, a factor that determines the fraction of the annualized spread applicable to this particular quarter.

Let $D_t(T - t)$ denote the appropriate money market discount factor starting from time t to some future date T. This discount factor is typically random. To deal with randomness in discount factor, default and recovery rate, we adopt the familiar derivatives pricing theory with which there exists a risk-neutral pricing measure, and derivatives can be priced by taking expectation of its contingent payment with respect to the risk-neutral pricing measure as if economic agents were not risk adverse. We denote such a risk-neutral expectation operator at time t by $E_t^Q(\cdot)$ to reflect the time-t information set.

I.1. CDS Par Spread and Actuarial Par Spread

The CDS par spread, $S_t(T - t)$, is known to satisfy the following condition, which reflects the fact that no money has changed hands initially between the protection buyer and seller.[1]

$$
E_t^Q[(1 - R_\tau)D_t(\tau - t)1_{\{t < \tau \leq t'_k\}}]
$$
$$
= S_t(T - t)\sum_{i=1}^{k}\{A(t_{i-1} \vee t, t'_i)
$$
$$
\times E_t^Q[D_t(t_i - t)1_{\{t'_i < \tau\}}]
$$
$$
+ E_t^Q[A(t_{i-1} \vee t, \tau)D_t(\tau - t)1_{\{t'_{i-1} < \tau \leq t'_i\}}]\},
$$
$$(1)$$

where $t_{i-1} \vee t$ denotes the maximum of t_{i-1} and t so that the partial accrual for the first payment period, i.e., from t_0 to t, is taken out. Note that the left-hand side is

the present value of the payment for the protection leg up to and including the final accrual end date, whereas the right-hand side is the present value of the payments for the premium leg with the amount determined by the CDS par spread that is known when the contract is entered. One needs to note that for all pre-scheduled payments, we have used the payment date, t_i, for discounting and the accrual end date, t'_i, for computing probability to reflect the actual default protection coverage period. Also implicit in the above equation is the assumption that at default, the payments for both premium and protection legs are made immediately. Any delay in payment can in effect be incorporated into the recovery rate.

Equation (1) can be used to state the CDS par spread as:

$$
S_t(T - t)
$$
$$
= \frac{E_t^Q\left[(1 - R_\tau)D_t(\tau - t)1_{\{t < \tau \leq t'_k\}}\right]}{\sum_{i=1}^{k}\left\{A(t_{i-1} \vee t, t'_i)E_t^Q\left[D_t(t_i - t)1_{\{t'_i < \tau\}}\right] + E_t^Q\left[A(t_{i-1} \vee t, \tau)D_t(\tau - t)1_{\{t'_{i-1} < \tau \leq t'_i\}}\right]\right\}}.
$$
$$(2)$$

Let $r_t(s, q)$ be the time-t risk-free forward discount rate starting at time $t + s$ with a duration of $q - s$, where $q \geq s$. The standard term structure theory implies that $r_t(0, T - t) = -\frac{1}{T-t}\ln(E_t^Q[D_t(T - t)])$. If we further assume that R_τ, τ and $D_t(t_i - t)$ (for all i's) are all independent and let $\bar{R}_t = E_t^Q(R_\tau)$, then

$$
S_t(T - t)
$$
$$
= \frac{(1 - \bar{R}_t)E_t^Q\left[e^{-r_t(0, \tau - t)(\tau - t)}1_{\{t < \tau \leq t'_k\}}\right]}{\sum_{i=1}^{k}\left\{A(t_{i-1} \vee t, t'_i)e^{-r_t(0, t_i - t)(t_i - t)}E_t^Q\left[1_{\{t'_i < \tau\}}\right] + E_t^Q\left[A(t_{i-1} \vee t, \tau)e^{-r_t(0, \tau - t)(\tau - t)}1_{\{t'_{i-1} < \tau \leq t'_i\}}\right]\right\}}.
$$
$$(3)$$

Now we are in a position to define the actuarial par spread, denoted by $S_t^{(a)}(T - t)$, which is the par spread computed by replacing the risk-neutral probability measure Q with the physical probability measure P. In other words, the actuarial par spread would become the CDS par spread when economic agents were truly

risk-neutral and valuation were conducted purely on an actuarial basis. That is to mean

$$S_t^{(a)}(T-t)$$

$$= \frac{(1-\bar{R}_t)E_t^P\left[e^{-r_t(0,\tau-t)(\tau-t)}1_{\{t<\tau\leq t_k'\}}\right]}{\sum_{i=1}^k\left\{A(t_{i-1}\vee t,t_i')e^{-r_t(0,t_i-t)(t_i-t)}E_t^P\left[1_{\{t_i'<\tau\}}\right]\right.}$$

$$\left.+\ E_t^P\left[A(t_{i-1}\vee t,\tau)e^{-r_t(0,\tau-t)(\tau-t)}1_{\{t_{i-1}'<\tau\leq t_i'\}}\right]\right\}}.$$

$$(4)$$

I.2. Term Structure of Physical PDs

In order to compute the actuarial spread, we need a model for term structure of PDs with respect to the physical probability law that governs occurrences of default. Like interest rates, term structure of PDs can be stated in terms of spot or forward PDs. Here, we adopt the forward intensity approach of Duan *et al.* (2012) to characterize term structure of PDs. Before proceeding further, it is important to note that a corporate exit can be caused by default/bankruptcy or simply merger/aquisition. However, exits for reasons other than default/bankruptcy are in the literature typically left un-modeled except for a few papers like Duffie *et al.* (2007), Duan *et al.* (2012) and Duan and Fulop (2013). The resulting censoring bias of course depends on the occurrence rate of other exits. Based on Table 1 of Duan *et al.* (2012) which documents defaults/bankruptcies and other types of firm exit for the US exchange-listed firms over 1991 to 2011, the censoring bias is likely significant; for example in 2007, 0.5% of the firms defaulted whereas 9.34% of them exited for other reasons.

This huge gap implies that ignoring other exits in default models could grossly inflate the predicted default probability, because it is the only factor for tuning both survival and default occurrences. Suppose p is the default probability for one period and the default rate is time homogeneous. Depending on how the data sample is treated, the consequences may differ. If the firms due to other exits are left out of the sample, p will be overestimated simply because the sample shows an upward biased default rate. If those firms are kept in the sample and treated as survived ones, p will be correctly estimated. Even if p is estimated properly,

there will be distorted consequences in applications; for example, the probability of a firm defaulting in two periods will be erroneously calculated as $p+(1-p)p$, but the correct PD should be lower because the right formula is $p+(1-p-q)p$ with q being the probability of other exits over one period. In the case of CDS, other exit does not mean termination because the successor entity will be assigned to replace the reference entity. Naturally, the relevant default probability must factor in the default probability of the successor; for example, qp^* should be added to $p+(1-p-q)p$ with p^* denoting the successor's default probability over the second period.

Envision a continuous-time setting where corporate obligors evolve dynamically with their individual attributes changing along with macroeconomic factors. Some obligors default, some exit for other reasons, but the rest survive. The occurrence of default and that of other exits for each obligor are governed by two Poisson processes that are independent when conditioning on their Poisson intensities. But these two intensities are correlated because they are functions of some common variables (obligor's attributes and macroeconomic factors). Across obligors, the default and other-exit processes are also independent once being conditioned on their Poisson intensities. Such processes are known as the Cox doubly stochastic processes, which serve as the modeling foundation for Duffie *et al.* (2007), Duan *et al.* (2012) and many others. The departing point for Duan *et al.* (2012) is the use of forward Poisson intensities to model the occurrence of default or other exits. Here, we simply apply the results directly and refer readers to Duan *et al.* (2012) for more technical details.

Let $f_t(s)$ and $h_t(s)$ be, respectively, the time-t forward default and other-exit intensities with the forward starting time of $t+s$. Define $\psi_t(s,q)=\frac{\int_s^q[f_t(u)+h_t(u)]du}{q-s}$ for $s\leq q$, which is in a way much like the standardized time-t forward interest rate starting at time $t+s$ with a duration of $q-s$. Recall our earlier definition for forward interest rates, $r_t(s,q)$. Note that the instantaneous forward interest rate starting at time $t+s$ can be expressed as $r_t(s,s)$, and it can be related to forward rates with longer durations by $r_t(s,q)=\frac{\int_s^q r_t(u,u)du}{q-s}$. In a pure sense of corporate default, the default probability over $[t,t_i']$ should

equal $\int_0^{t_i'-t} e^{-\psi_t(0,u)u} f_t(u)du$ and the discounted default probability would be $\int_0^{t_i'-t} e^{-[r_t(0,u)+\psi_t(0,u)]u} f_t(u)du$. However, they are not the right default and discounted default probabilities for CDS, because substitution typically takes place when the reference entity is merged with or acquired by another corporate entity. In that case, the CDS protection is typically shifted to the merged or acquiring entity, and the successor entity will then face subsequent default or other exit. Therefore, one must be careful in applying a suitable treatment.

Let $P_t(s, q; r_t(0, u), s \le u \le q)$ denote the time-t discounted forward probability of the reference entity of the CDS being terminated, including successions, over the period $[t + s, t + q]$, where $0 \le s \le q$. This termination probability is meant to capture an intricate sequence of potential defaults by the original reference entity and its successors. This probability ultimately determines the likelihood of the CDS seller having to make a default payment. Similarly, $P_t^*(s, q; r_t(0, u), s \le u \le q)$ stands for the termination probability of the successor. When there is no ambiguity, we will simply express these discounted forward termination probabilities as $P_t(s, q)$ and $P_t^*(s, q)$. What we mean by discounted probability is the integration of the forward probability rate at, say, time $t + u$ (with $s \le u \le q$) being discounted by the time-t forward risk-free rate from time $t + s$ to time $t + u$. The related quantities for the successor are similarly denoted by $f_t^*(s), h_t^*(s)$ and $\psi_t^*(s, q)$. Let $1_{\{Sub\}}$ be an indicator with 1 for the CDS contract that is subject to substitution and 0 otherwise.

The succession entity of course cannot be determined beforehand. Therefore, some reasonable rules will be needed for implementation. One can, for example, assume the successor to be an entity that shares the same forward termination probability as the reference obligor, or an entity that is of median quality. In principle, succession may occur multiple times, which in turn requires us to make a further assumption on all subsequent substitutions. For analytical tractability, our maintained assumption is that the successor's forward termination probability may differ from that of the original reference entity, but all successors must be of the same type in terms of the forward termination probability. We show in Appendix A that the solution

to the CDS discounted forward termination probability with or without substitution for $0 \le s \le q$ can be expressed as

$$P_t(s, q) = \int_s^q e^{-[r_t(s,u)+\psi_t(s,u)](u-s)} f_t(u)du$$
$$+ \int_s^q e^{-[r_t(s,u)+\psi_t(s,u)](u-s)} h_t(u)$$
$$\times \left[1_{\{Sub\}} P_t^*(u, q) + 1 - 1_{\{Sub\}} \right] du, \quad (5)$$

where

$$P_t^*(s, q) = \int_s^q e^{-\int_s^v [r_t(u,u)+f_t^*(u)]du} f_t^*(v)dv. \quad (6)$$

In general, formulas in (5) and (6) can be used repeatedly to compute the following quantity in the first term of the denominator of Eq. (4) by temporarily setting forward interest rates to zero:

$$E_t^P \left(1_{\{t_i'<\tau\}} \right) = 1 - P_t(0, t_i' - t; r_t(0, u) = 0$$
$$\text{for } 0 \le u \le t_i' - t). \quad (7)$$

When the substitution feature is disabled as in Case 2 of Appendix B, the above quantity naturally reduces to $\exp[-\psi_t(0, t_i' - t)(t_i' - t)]$, a standard result for the probability of surviving default and other exits over the period $[t, t_i']$.

The solutions to the two remaining terms in Eq. (4) again depend on $P_t^*(s, q)$ in Eq. (6). They can be expressed as

$$E_t^P \left[e^{-r_t(0,\tau-t)(\tau-t)} 1_{\{t<\tau\le t_k'\}} \right]$$
$$= \int_0^{t_k'-t} e^{-[r_t(0,s)+\psi_t(0,s)]s} f_t(s)ds$$
$$+ \int_0^{t_k'-t} e^{-[r_t(0,s)+\psi_t(0,s)]s} h_t(s) 1_{\{Sub\}}$$
$$\times P_t^*(s, t_k' - t)ds \quad (8)$$

and

$$E_t^P \left[A(t_{i-1} \vee t, \tau)e^{-r_t(0,\tau-t)(\tau-t)} 1_{\{t_{i-1}'<\tau\le t_i'\}} \right]$$
$$= \int_{t_{i-1}'\vee t}^{t_i'} A(t_{i-1} \vee t, s)e^{-[r_t(0,s-t)+\psi_t(0,s-t)](s-t)}$$
$$\times f_t(s - t)ds$$

$$+ \int_{t'_{i-1} \vee t}^{t'_i} A(t_{i-1} \vee t, s) e^{-[r_t(0,s-t)+\psi_t(0,s-t)](s-t)}$$

$$\times h_t(s-t) 1_{\{Sub\}} P_t^*(s-t, t'_i - t) ds. \tag{9}$$

Note that forward interest rate, $r_t(s, u)$, used in Eqs. (5) and (6) can always be deduced from the spot interest rate curve.

In Appendix B, we provide the specific solutions for two special cases of interest. One of which (with substitution and its successors sharing the same forward intensities as the original obligor) will be used for the numerical example and empirical study later because substitution is a built-in feature of standard CDS contracts.[2]

The actuarial par spread formula can be readily computed daily by leveraging the Credit Research Initiative (CRI) infrastructure created and maintained by the Risk Management Institute (RMI) of the National University of Singapore since July 2010. The RMI-CRI system has implemented the forward intensity model of Duan *et al.* (2012). It currently makes available and allows free access by all legitimate users to daily updated PDs ranging from one month to five years for over 60,000 exchange-listed firms in 106 economies around the world. The forward intensity functions used to generate the RMI-CRI PDs are exponential linear functions of some input variables (2 macroeconomic factors and 10 firm-specific attributes) where the coefficients depend on the forward starting time, and are subject to the Nelson–Siegel type of smooth term structure restriction. Estimation of the parameters and statistical inference for the constrained system rely on the pseudo-Bayesian sequential Monte Carlo technique and self-normalized statistics devised in Duan and Fulop (2013). For details on the specific RMI-CRI implementation, please refer to RMI-CRI Technical Report (2013). The RMI-CRI model's parameters are re-calibrated monthly and the inputs to the functions are updated daily.

II. A NUMERICAL EXAMPLE

We will compute a 5-year actuarial par spread by daily discretization and the convention of quarterly

payments subject to the actual/360 day count. Thus, all integrals above are just some simple sums with the term, ds, being approximated by $1/365$. $A(t_{i-1} \vee t, t'_i)$ in the formula is simply the fraction of a year defined by the actual number of days in an accrual period (a quarter or shorter) over 360, which reflects the CDS day count convention. It should be noted that for all accrual periods, $A(t_{i-1} \vee t, t'_i)$ or $A(t_{i-1} \vee t, \tau)$, should be inclusive of the start and end dates.

For the numerical example and the empirical analysis later, we assume that the successor to replace the reference entity has the same discounted forward termination probability, i.e., $P_t^*(s, T - t) = P_t(s, T - t)$ for $0 \le s \le T - t$. The relevant formulas are given in Case 1 of Appendix B. Now, we are in a position to show how to compute the actuarial par spread using the 5-year CDS referencing Eastman Kodak on November 16, 2011. Note that Eastman Kodak filed for Chapter 11 bankruptcy protection on January 19, 2012. The CDS traded at 4,009.84 basis points (bps) on November 16, 2011. Our calculation later shows that the actuarial par spread is estimated to be only 422.66 bps based on the standard recovery rate of 40%. Assuming that the term structure of PDs and the recovery assumption used in our analysis are reasonably accurate, the huge gap of 3,587.17 bps could only be attributed to a combination of very high risk aversion, very low market liquidity (extremely tight supply condition) and an overestimation by the market of the PDs. It is also possible that the huge gap is due to an underestimation of the PDs by the physical default prediction model and/or using an overstated recovery rate.

The typical assumption of a 40% recovery rate may indeed be too optimistic, leading to an understated actuarial par spread.[3] But even using the extreme case of zero recovery, the actuarial par spread would still be just around 700 bps. The point of this discussion is not about whether a particular estimate is correct or not; rather the actuarial par spread can be a useful device for decomposing the observed CDS spread so as to help us gain better understanding of the CDS market behavior.

The 5-year CDS traded on November 16, 2011 has its maturity on December 20, 2016 and potentially up to 21 quarterly premium payments on 20th of March/June/September/December if the 20th is a

business day and otherwise the first business day after that 20th.[4] For the last payment date, however, the adjustment is capped at 20th plus at most two calendar days. Since December 20, 2016 is a business day, no adjustment is needed. The first premium payment date is the closest coupon payment date after the trade date plus one day, and in this case the first payment date is December 20, 2011. Reported in Table 1 are the actual payment dates (after adjusting the official payment dates) for this CDS. These dates of course determine the applicable length of time and discount rate for the discounting purpose. However, they differ from the accrual end dates by one calendar day except for the final payment date.

The accrual period for the first payment starts on September 20, 2011, which is the closest standard quarterly payment date prior to the first payment date of this 5-year CDS. Subsequent accrual periods are spaced

Table 1. The 5-year CDS payment dates, accrual periods, and number of days in each period for the trade date of November 16, 2011.

Payment #	Payment Date	Accrual Start Date (inclusive)	Accrual End Date (inclusive)	# of Days
1	20.12.2011	20.09.2011	19.12.2011	33[a]
2	20.03.2012	20.12.2011	19.03.2012	91
3	20.06.2012	20.03.2012	19.06.2012	92
4	20.09.2012	20.06.2012	19.09.2012	92
5	20.12.2012	20.09.2012	19.12.2012	91
6	20.03.2013	20.12.2012	19.03.2013	90
7	20.06.2013	20.03.2013	19.06.2013	92
8	20.09.2013	20.06.2013	19.09.2013	92
9	20.12.2013	20.09.2013	19.12.2013	91
10	20.03.2014	20.12.2013	19.03.2014	90
11	20.06.2014	20.03.2014	19.06.2014	92
12	22.09.2014	20.06.2014	21.09.2014	94
13	22.12.2014	22.09.2014	21.12.2014	91
14	20.03.2015	22.12.2014	19.03.2015	88
15	22.06.2015	20.03.2015	21.06.2015	94
16	21.09.2015	22.06.2015	20.09.2015	91
17	21.12.2015	21.09.2015	20.12.2015	91
18	21.03.2016	21.12.2015	20.03.2016	91
19	20.06.2016	21.03.2016	19.06.2016	91
20	20.09.2016	20.06.2016	19.09.2016	92
21	20.12.2016	20.09.2016	20.12.2016	92

Note: [a]This value reflects the number of days in the first accrual period that is applicable to this CDS contract, i.e., from November 17, 2011 (trade date plus one day) to December 19, 2011 inclusive.

according to the official payment dates with the accrual period ending one calendar day prior to the payment date except for the last payment period which actually ends on the official maturity date. Naturally, these accrual periods are not exactly identical in the number of calendar days, and the number of days will have to be computed according to the business day calendar. For this particular 5-year CDS, the accrual period and the number of calendar days in each period are given in Table 1. The number of days as reported in Table 1 for each period divided by 360 should be used for $A(t_{i-1} \vee t, t_i')$ in Eq. (4). For $A(t_{i-1} \vee t, \tau)$ in the same equation, it is the actual number of days elapsed (inclusive of the default day) and then divided by 360; for example, default on January 19, 2012 means an accrual period of 41 days (from December 20, 2011 to January 19, 2012 inclusive).

As mentioned in the preceding section, daily updated PD term structures for nearly all exchange-listed firms around the world are currently provided by RMI. We extract Eastman Kodak's PD term structure on November 16, 2011. RMI computes all PDs using data obtained after each market close. Data preparation and computation take several hours and the PDs become publicly available on the next calendar day. For a small set of firms, however, the PDs can be computed on a real time basis with access to, say, a Bloomberg terminal. Our use of the PD information on the trade date is actually operationally feasible.

There are two sets of forward intensity functions at time t (i.e., November 16th), $f_t(s)$ for default and $h_t(s)$ for other exits where s stands for forward starting time from t. The set of prediction variables (covariates) used by RMI for both default and other exits comprises, as of the time of this writing, 12 variables (2 macro-financial and 10 firm-specific) and their values are based on the most current and publicly available information. In our case, it is at the market close of November 16, 2011. The variables and their values on November 16, 2011 for Eastman Kodak are provided in Table 2, which are based on the March 2014 calibration by RMI.

The forward intensity functions implemented by RMI-CRI are in the following form:

$$f_t(s) = \exp\{\alpha_0(s) + \alpha_1(s)x_{1,t} + \cdots + \alpha_{12}(s)x_{12,t}\}, \qquad (10)$$

$$h_t(s) = \exp\{\beta_0(s) + \beta_1(s)x_{1,t} \\ + \cdots + \beta_{12}(s)x_{12,t}\}. \quad (11)$$

The coefficients as functions of forward starting time are constrained by the Nelson–Siegel function of four or three parameters, depending on whether the covariate is stochastic or not. For the coefficients in the default intensity function, the RMI-CRI system, as of the time of this writing, imposes

$$\alpha_i(s; \varrho_{i,0}, \varrho_{i,1}, \varrho_{i,2}, d_i)$$
$$= \varrho_{i,0} + \varrho_{i,1}\frac{1 - \exp(-s/d_i)}{s/d_i}$$
$$+ \varrho_{i,2}\left[\frac{1 - \exp(-s/d_i)}{s/d_i} - \exp(-s/d_i)\right]$$
$$\text{for } i = 0, 1, 2, \ldots, 12 \text{ and } \varrho_{i,0} = 0$$
$$\text{for } i = 1, 2, \ldots, 12. \quad (12)$$

Similar constraints are placed on the coefficients of the other-exit intensity function. The relevant parameter values needed to obtain the PDs for American firms with the trade date of November 16, 2011 are given in Table 3.[5]

Table 2. The variables (covariates) used in the forward intensity functions (default and other exits) and their values on November 16, 2011 extracted from the RMI-CRI database as of February 28, 2014, which may reflect data revisions post November 16, 2011. Trend stands for the current value minus one-year moving average. DTD is distance to default which is an estimate based on a structural credit risk model. Please refer to RMI-CRI Technical Report (2013) for details on this measurement along with other variables.

Prediction Variable	Value
S&P500 Index Return (1 year)	0.0497
3-month US Treasury Rate (demeaned)	−3.1928
DTD (Level)	−0.0288
DTD (Trend)	−0.6534
Cash/Total Assets (Level)	0.2163
Cash/Total Assets (Trend)	−0.0369
Net income/Total Assets (Level)	−0.0146
Net income/Total Assets (Trend)	0.0035
Relative Size (Level)	0.4176
Relative Size (Trend)	−0.6963
Market/Book	1.3303
Sigma	0.3236

The parameter values in Table 3 allow us to compute, for example, the probability of not defaulting before the first premium payment date of December 20, 2011. Daily discretization is employed, which means $\Delta s = 1/365$. This discretization matches well with the operational reality of counting days. The integral in Eq. (B.1) thus becomes a sum of 33 terms corresponding to 33 calendar days between November 16, 2011 and December 19, 2011. By Eq. (B.2), we have

$$E_{11/16/2011}^P\left(1_{\{12/19/2011<\tau\}}\right)$$
$$= 1 - P_{11/16/2011}(0, 33\Delta s)$$
$$= 1 - \Delta s \sum_{i=0}^{32} f_{11/16/2011}(i\Delta s)$$
$$\times \exp\left(-\Delta s \sum_{j=0}^{i} f_{11/16/2011}(j\Delta s)\right).$$

The values for the forward intensities in the above expression can be computed with Eqs. (10)–(12) and the values in Tables 2 and 3.

Several quantities in Eq. (4) for the actuarial par spread need an appropriate interest rate term structure. The standard practice in pricing CDS is to use the interest rate term structure extracted from a combination of LIBOR rates and Swap rates by a bootstrap technique. The missing rates are set by linearly interpolating the available interest rates (in continuously compounded form). Readers can refer to Markit (2013) for details of constructing such a term structure. Table 4 provides the LIBOR and Swap rates on November 16, 2011 that were retrieved from Bloomberg.[6] Since daily discretization is used, we have to compute the discount rates for maturities ranging from one calendar day (November 17, 2011) to 1861 calendar days (December 20, 2016) with daily increments. The one-day continuously compounded rate by applying the LIBOR day count convention turns out to be 0.1416% whereas the 1861-day continuously compounded rate is 1.3441%. With the term structure of interest rates in place, one can proceed to compute forward interest rates (different forward starting times but always with one-day duration) needed in the formulas.

Other formulas needed to compute the actuarial par spread are in Eqs. (B.3) and (B.4). With the term

Table 3. The parameter values used to determine the forward intensity functions for the listed American firms for the trade date of November 16, 2011.

Variable #	$\varrho_{i,0}$	$\varrho_{i,1}$	$\varrho_{i,2}$	d_i
Default Intensity Function				
Intercept	−4.8004	1.2312	3.8558	0.6316
S&P500 Index Return (1 year)	0	−0.0538	90.3545	88.2970
3-month US Treasury Rate (demeaned)	0	−7.5111	47.1087	0.9615
DTD (Level)	0	−1.2931	−0.0168	0.8524
DTD (Trend)	0	−1.0767	−0.0106	0.9322
Cash/Total Assets (Level)	0	−0.9887	0.9456	0.5911
Cash/Total Assets (Trend)	0	−2.1815	0.3087	2.8732
Net income/Total Assets (Level)	0	−4.2162	−10.9831	7.7829
Net income/Total Assets (Trend)	0	−1.1745	0.2412	8.1014
Relative Size (Level)	0	−0.0288	0.2506	0.5198
Relative Size (Trend)	0	−1.7689	2.5739	1.9547
Market/Book	0	−0.0424	0.0679	0.6598
Sigma	0	−0.1883	−137.2612	2570.6303
Other-Exit Intensity Function				
Intercept	−2.4898	−0.6837	0.5232	1.1805
S&P500 Index Return (1 year)	0	−0.1455	2.4191	0.8112
3-month US Treasury Rate (demeaned)	0	0.8581	5.4323	1.3832
DTD (Level)	0	0.0214	−0.2274	1.9629
DTD (Trend)	0	0.2734	−0.4832	0.3040
Cash/Total Assets (Level)	0	−0.3705	1.5446	1.4461
Cash/Total Assets (Trend)	0	−0.6143	0.9736	6.5312
Net income/Total Assets (Level)	0	−3.9779	5.3066	5.2265
Net income/Total Assets (Trend)	0	−2.3135	0.0711	0.4868
Relative Size (Level)	0	−0.1781	−0.1221	4.3556
Relative Size (Trend)	0	−0.5261	1.5800	11.5417
Market/Book	0	−0.0290	0.0722	4.5704
Sigma	0	1.9060	−1.4790	0.9276

Table 4. LIBOR and Swap rates (in %) on November 16, 2011.

LIBOR		Swap	
Tenor	Rate	Tenor	Rate
1 D	0.14167	2 Y	0.7590
1 W	0.19189	3 Y	0.8646
2 W	0.21039	4 Y	1.0697
1 M	0.25172	5 Y	1.3180
2 M	0.35706	6 Y	1.5623
3 M	0.47111		
4 M	0.54417		
5 M	0.60722		
6 M	0.67694		
7 M	0.73278		
8 M	0.78100		
9 M	0.83222		
10 M	0.88322		
11 M	0.93744		
1 Y	0.99700		

structure of interest rates and the intensity functions in place, the numerator of Eq. (4) is computed to be 0.1670 and the two terms in the denominator are 3.9204 and 0.0296, respectively. Thus, the actuarial par spread becomes 0.042266 (or 422.66 bps).

III.   EMPIRICAL PRICING OF CDS VIA DECOMPOSITION

CDS spreads have been used in the literature as an instrument to understand corporate bond yields; for example, Longstaff *et al.* (2005) and Li *et al.* (2011), among others. The literature on the theoretical and empirical pricing of CDS is quite large. Our focus in this section is to provide an easy-to-implement empirical pricing tool for CDS through a decomposition with

the actuarial par spread. Our approach differs substantially from, say, Ericsson *et al.* (2009) in which CDS is related to leverage, volatility and interest rate. Although these are theoretical determinants of CDS price, they are indirect and incomplete. They are indirect because although leverage and volatility do affect an obligor's default likelihood, they are not as close to it as the physical default probability. They are also incomplete because CDS price is expected to be closely tied to recovery rate and risk aversion.

We use a time series of Eastman Kodak CDS to demonstrate the idea of our empirical pricing through decomposition. A year-long time series of daily CDS spreads, obtained from Bloomberg and exhibited in Fig. 1, shows a dramatic rise of CDS premiums leading up to its Chapter 11 bankruptcy filing on January 19, 2012. Of course, this rising pattern is not surprising in light of Kodak's subsequent default. In the same figure, we plot the actuarial par spread over the same time period. Generally speaking, the two time series move in tandem although the CDS spread is much larger than the corresponding actuarial par spread.[7] This cursory evidence points to the possibility of establishing some empirical relationship between these two quantities.

In Fig. 2, we plot the log-ratio of the CDS spread over its corresponding actuarial par spread. As the figure shows, the log spread ratio hovers around its mean of 2.0867 with a standard deviation of 0.2968. Moreover, we can compute its skewness and excess kurtosis, and their values of 0.0953 and −0.2369 seem to suggest that the log spread ratio is not far from being normally distributed. A quick way of pricing CDS is to first look up its actuarial par spread and add on top an amount that reflects the mean of 2.0867 for the log spread ratio. The mean log spread ratio is likely to be firm-specific. For the CDS without a sufficiently long history, one could use a cross-sectional sample of mean log spread ratios corresponding to different obligors to come up with a suitable estimate.

The plot also suggests some degree of autocorrelation, and with autocorrelation one may actually come up with a workable prediction model for the log spread ratio. Indeed, our analysis suggests that a simple lagged regression can yield a high R^2 of 85% and give rise to the following predictive equation:

$$\ln\left(\frac{S_t}{S_t^{(a)}}\right) \approx 0.1487 + 0.9296 \times \ln\left(\frac{S_{t-1}}{S_{t-1}^{(a)}}\right). \quad (13)$$
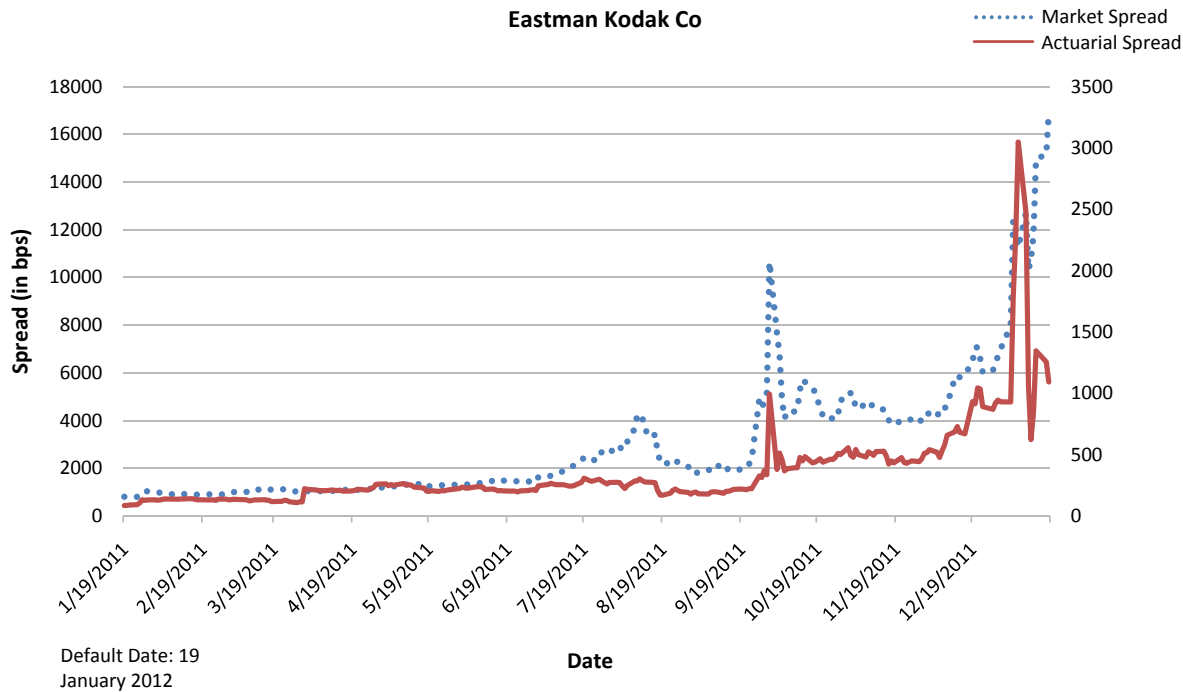

Figure 1. One-year daily time series of 5-year CDS spreads (the left axis) and actuarial par spreads (the right axis) for Eastman Kodak.
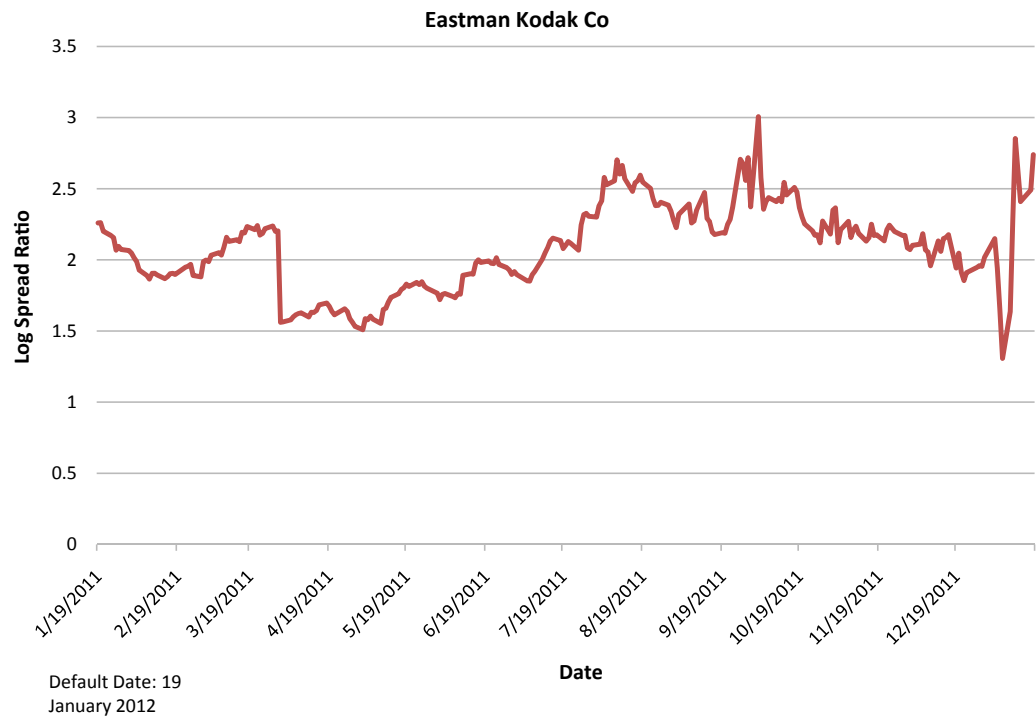
Figure 2. Time series of the log-ratio of the 5-year CDS spreads over their corresponding actuarial par spreads for Eastman Kodak.
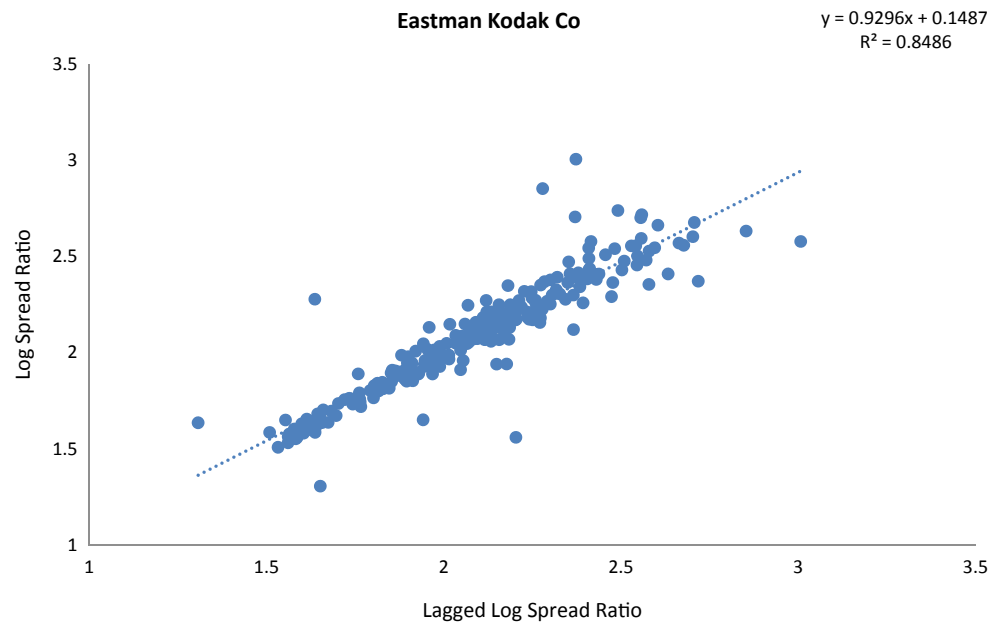

Figure 3. The log-ratio of the 5-year CDS spread over its corresponding actuarial par spread versus the log-ratio being lagged one trading day.

In Fig. 3, the visual relationship is clearly reflective of the above lagged regression. With the exception of a few scattered points, the log spread ratio seems to be tightly centered around the regression line. For pricing CDS, the predicted log spread ratio based on Eq. (13), instead of the unconditional mean of 2.0867, can be applied to produce a better estimate of CDS spread that is in principle much lower in pricing error. If we were to price the CDS on the trade date of November 16, 2011 using the actuarial par spread of 422.66 and the average log spread ratio of 2.0867, we would obtain a predicted CDS spread of 3406.03 bps (i.e., $422.66 \times \exp(2.0867)$). Alternatively, we could price this CDS by its predicted log spread ratio of 2.1522 computed with the log spread ratio of 2.1552 on November 15, 2011 (CDS spread $= 4228.85$ bps and actuarial par spread $= 490.01$ bps). Doing so gives rise to a predicted CDS spread of 3636.56 bps (i.e., $422.66 \times \exp(2.1522)$). As compared to 3406.03 bps, this alternative predicted spread is 230.53 bps closer to the observed CDS spread of 4009.84 bps on November 16, 2011.

In the above example, we considered the CDS pricing at the end of the trade date. Actually, empirical pricing can be performed throughout the day as the relevant information (stock price, LIBOR curve, etc.) arrives at the trading desk. While the parameter values of the PD model remains fixed, the input values to the PD model change according to the new information. The revised default and other-exit probabilities then lead to a new actuarial par spread which in turn yields a new estimate of the CDS spread through the same predicted log spread ratio.

ACKNOWLEDGMENT

NOTES

[1] The CDS pricing discussed here is quite standard; see for example, Hull (2012, Chapter 24). Later when we compute with survival and default probabilities, we will depart from the standard model to a more appropriate one that factors in a corporate obligor's exit due to reasons other than default (say, merger/acquistion).

[2] This case essentially implies that all successors are firms of the same forward termination probability as that of the original obligor, which is a sensible assumption and simplifies the implementation task considerably. It is sensible because there is no way of predetermining which firm will end up being the successor several years before an acquisition. Having a successor with the same quality on a forward basis as the original obligor seems to be a good compromise. Operationally speaking, it is more manageable because the calculation can be completed without referencing the forward termination probability of another firm.

[3] Recovery rates and PDs may also be correlated, which has been ruled out by an assumption earlier. With a suitable conditional recovery rate model, it is possible to incorporate this correlation into the actuarial par spread formula.

[4] See ISDA (2012).

[5] The specific parameter values in Table 3 are taken from the RMI-CRI March 2014 calibration which used its database as of February 28, 2014. Thus, the parameter values have naturally reflected some data revisions post November 16, 2011, and the model calibration has also used some data that were beyond November 16, 2011.

[6] According to the ISDA standard CDS converter specification (ISDA, 2009), the interest rate curve used in the CDS discounting is locked on the trade date minus one day. Note that the ISDA specification is for the purpose of quotation conversion. Fixing the discount curve from a day earlier can avoid confusion arising from different discount curves in a trading day. Here, we choose to use the curve on the trade date instead, because it is actually operationally feasible to use updated information throughout the trade day for the purpose of CDS pricing. Typically, the interest curve does not change drastically over one day, and it will thus make no material difference for a typical day.

[7] In the final month, i.e., January 2012, Kodak's actuarial par spreads have swung widely. In light of the CDS behavior over the same period, intuition leads one to question the quality of the physical PDs used to produce those actuarial par spreads. One possible reason is our use of extrapolation to produce default and other-exit probabilities for periods shorter than one month. Since the RMI-CRI PD model is calibrated to data by predictions ranging from 1 to 60 months, any prediction for a horizon less than one month must use the extrapolated parameter values obtained via the functional restriction in Eq. (12). When an obligor approaches default, shorter-term PDs naturally play a more prominent role in determining actuarial par spreads. Small random changes to the input data (such as DTD, size, etc.) over Kodak's waning days get magnified through extrapolated model parameters. A potential fix is to run calibration by incorporting shorter-period predictions to avoid using extrapolation.

APPENDIX

Appendix A

The discounted forward termination probability for a CDS reference entity can be written as

$$P_t(s, q) = \int_s^q e^{-[r_t(s,u)+\psi_t(s,u)](u-s)} f_t(u)du$$

$$+ \int_s^q e^{-[r_t(s,u)+\psi_t(s,u)](u-s)} h_t(u)$$

$$\times \left[1_{\{Sub\}} P_t^*(u, q) + 1 - 1_{\{Sub\}} \right] du.$$

The first term on the right-hand side is the typical time-t cumulative discounted forward probability of termination due to default from time $t + s$ to $t + q$, whereas the second term is the one for other-exits over the same period but modified by the discounted forward termination probability of the successor entity, if substitution is in force. This modification reflects default by the successor after it replaces the reference entity.

To obtain a solution to the forward termination probability of the successor, we assume that substitution is in force. In addition, the successor and all its subsequent successors are required to share the same forward termination probability, but this forward termination probability can differ from that of the original reference entity. Under these assumptions, we have

$$P_t^*(s, q)$$

$$= \int_s^q e^{-[r_t(s,u)+\psi_t^*(s,u)](u-s)} f_t^*(u)du$$

$$+ \int_s^q e^{-[r_t(s,u)+\psi_t^*(s,u)](u-s)} h_t^*(u) P_t^*(u, q)du.$$

Taking a partial derivative with respect to s gives rise to the following linear first-order differential equation:

$$\frac{\partial P_t^*(s, q)}{\partial s}$$

$$= -f_t^*(s) + [r_t(s, s) + f_t^*(s) + h_t^*(s)]$$

$$\times \int_s^q e^{-[r_t(s,u)+\psi_t^*(s,u)](u-s)} f_t^*(u)du$$

$$- h_t^*(s) P_t^*(s, q) + [r_t(s, s) + f_t^*(s) + h_t^*(s)]$$

$$\times \int_s^q e^{-[r_t(s,u)+\psi_t^*(s,u)](u-s)} h_t^*(u) P_t^*(u, q)du$$

$$= -f_t^*(s) + [r_t(s, s) + f_t^*(s)] P_t^*(s, q).$$

Recall that $r_t(s, s)$ is the time-t instantaneous forward rate for time $t + s$. Naturally, this differential equation must obey the terminal condition: $P_t^*(q, q) = 0$.

The general solution to this linear first-order differential equation is

$$P_t^*(s, q) = e^{-\int_s^q [r_t(u,u)+f_t^*(u)]du}$$

$$\times \left(\int_s^q e^{\int_v^q [r_t(u,u)+f_t^*(u)]du} f_t^*(v)dv + C \right).$$

By the terminal condition, we have $C = 0$. Thus,

$$P_t^*(s, q) = e^{-\int_s^q [r_t(u,u)+f_t^*(u)]du}$$

$$\times \int_s^q e^{\int_v^q [r_t(u,u)+f_t^*(u)]du} f_t^*(v)dv$$

$$= \int_s^q e^{-\int_s^v [r_t(u,u)+f_t^*(u)]du} f_t^*(v)dv.$$

Appendix B

Case 1. Substitution is in force, i.e., $1_{\{Sub\}} = 1$, and the successor entity is assumed to always share the same default and other-exits intensities of the reference entity, i.e., $P_t^*(s, q) = P_t(s, q)$.

Under the conditions and applying the result in Appendix A, we obtain

$$P_t(s, q) = \int_s^q e^{-\int_s^v [r_t(u,u) + f_t(u)]du} f_t(v)dv. \qquad (B.1)$$

The relevant formulas for the actuarial par spread are

$$E_t^P \left(1_{\{t_i' < \tau\}}\right) = 1 - P_t(0, t_i' - t; r_t(0, u) = 0$$
$$\text{for } 0 \leq u \leq t_i' - t) \qquad (B.2)$$

and

$$E_t^P \left[e^{-r_t(0, \tau - t)(\tau - t)} 1_{\{t < \tau \leq t_k'\}}\right]$$
$$= \int_0^{t_k' - t} e^{-[r_t(0,s) + \psi_t(0,s)]s} f_t(s)ds$$
$$+ \int_0^{t_k' - t} e^{-[r_t(0,s) + \psi_t(0,s)]s} h_t(s) P_t(s, t_k' - t)ds$$
$$= P_t(0, t_k' - t). \qquad (B.3)$$

The last equality in the above is due to the result in Appendix A. In numerical evaluations of this paper, we actually use the first equality so as to be consistent with the expression below. Daily discretization used in the numerical evaluation can cause a minor difference between the first equality and the final expression.

The final quantity of interest is

$$E_t^P \left[A(t_{i-1} \vee t, \tau) e^{-r_t(0, \tau - t)(\tau - t)} 1_{\{t_{i-1}' < \tau \leq t_i'\}}\right]$$
$$= \int_{t_{i-1}' \vee t}^{t_i'} A(t_{i-1} \vee t, s) e^{-[r_t(0,s-t) + \psi_t(0,s-t)](s-t)}$$
$$\times f_t(s - t)ds$$
$$+ \int_{t_{i-1}' \vee t}^{t_i'} A(t_{i-1} \vee t, s) e^{-[r_t(0,s-t) + \psi_t(0,s-t)](s-t)}$$
$$\times h_t(s - t) P_t(s - t, t_i' - t)ds. \qquad (B.4)$$

Note that we cannot remove $h_t(s)$, the forward intensity for other exits, from the system in the way like for the other expression. This is simply due to the presence of $A(t_{i-1} \vee t, s)$, the day-count factor.

Case 2. No substitution takes place, i.e., $1_{\{Sub\}} = 0$, and the CDS contract terminates when exit for reasons other than default occurs. In this case, $P_t^*(s, q)$ becomes irrelevant.

Under the conditions, we obtain

$$P_t(s, q) = 1 - e^{-[r_t(s,q) + \psi_t(s,q)](q-s)}. \qquad (B.5)$$

The relevant formulas for the actuarial par spread are

$$E_t^P \left(1_{\{t_i' < \tau\}}\right) = 1 - P_t(0, t_i' - t; r_t(0, u) = 0$$
$$\text{for } 0 \leq u \leq t_i' - t), \qquad (B.6)$$

$$E_t^P \left[e^{-r_t(0, \tau - t)(\tau - t)} 1_{\{t < \tau \leq t_k'\}}\right]$$
$$= \int_0^{t_k' - t} e^{-[r_t(0,s) + \psi_t(0,s)]s} f_t(s)ds \qquad (B.7)$$

and

$$E_t^P \left[A(t_{i-1} \vee t, \tau) e^{-r_t(0, \tau - t)(\tau - t)} 1_{\{t_{i-1}' < \tau \leq t_i'\}}\right]$$
$$= \int_{t_{i-1}' \vee t}^{t_i'} A(t_{i-1} \vee t, s) e^{-[r_t(0,s-t) + \psi_t(0,s-t)](s-t)}$$
$$\times f_t(s - t)ds. \qquad (B.8)$$

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