Dynamic Macro Scenario Analysis via Bridge Sampling

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Abstract

Dynamic macroeconomic models should by design be amenable to scenario analyses under some stated policy objectives or presumed stress environment. However, such analyses are difficult to conduct mainly due to the technical complexity associated with partially conditioning on a future scenario. In this paper, we devise a generic and efficient bridging sampling method that is flexible with scenario setting while accommodating parameter uncertainty. We demonstrate that models with comparable forecasting performance can have very different implications under a policy scenario. Dynamic scenario analysis thus provides a new angle for discriminating competing models with a specific reference to real-world usage.

Keywords: sequential Monte Carlo, density tempering, parameter uncertainty, Gaussian models, inflation targeting, conditional forecast.

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1 Introduction

Scenario analysis in this paper refers to the study of the set of possible paths in the probability sense leading towards a targeted or hypothetical event/scenario. Since its introduction by Herman Kahn back in 1950s, scenario analysis has long been a popular strategic planning technique and alternative approach to business forecasting. According to an early survey of Linneman and Klein (1983), the use of scenario analysis among Fortune 100 corporates had increased rapidly from about 10% in 1974 to 50% by 1981. The topic itself also attracts continuous interest and attention from academic researchers. In Huss (1988), the author presents a comprehensive discussion on the weaknesses of conventional forecasting exercise that can be potentially addressed by scenario analysis. For instance, instead of providing confidence bands around a single view of the future which is typically the case with conventional forecasting, scenario analysis can be applied to generate completely different but plausible future situations to enable contingency planning and provide possible early warnings. Subsequent research in business management literature has been devoted to refine the scenario construction and prediction process, but mostly in qualitative sense.

Despite the various advantages scenario analysis has over conventional forecasting exercises, it does not receive as much attention as it deserves in macroeconomic studies, especially in policy analysis. Modern macroeconomic research focuses on two mainstream applications in producing unconditional forecasts and structural (shock) analysis. However, both applications have limitations on the extent to which they can directly address policy issues, and naturally fail to meet the needs of policymakers. Shock analysis through the study of impulse responses delivers meaningful economic interpretation regarding model dynamics, which is helpful in understanding the interrelationship among macro variables. In terms of giving instructive policy guidance, however, it is of little practical use as shocks are mostly unobserved and hard to quantify. From time to time, forecast is carried out by policymakers to serve the interest of predicting the future, and a great deal of effort has been devoted by researchers to improving economic forecasting accuracy. However, it is of limited use to policy makers as their objective is to shape the future, instead of merely knowing it.

As pointed out by Adolfson et al. (2007), for macro models to be used in providing basis for policy decisions, their benefits have to be clearly demonstrated in situations that policy advisors recognize. When policymakers initiate a new policy, they have an objective in mind which may or may not be publicly stated. For example, maintaining price stability is one of the most commonly recognized macroeconomic objectives. It has been adopted by major developed economies including the United States (since January 2012) and Japan (since January 2013). To justify their 2% inflation target, central bankers need to be clearly informed about its consequences. In other words, they look for an analysis which can tell them what could happen or how an economy would look like when they move from current state to the target scenario. It is here that scenario analysis can play an important role in addressing directly the macro objective and in providing a basis for subsequent decisions made. Furthermore, such analysis also provides meaningful insights for non-policymakers like firms and general public, to help them interpret information and form expectations based on the announced policy objective. After an objective has been set, there may be a set of alternative adjustment tools to choose from. Conducting so called 'policy simulation' is appealing to policymakers in understanding the pros and cons of each tool, and scenario analysis provides a natural tool for doing so. In fact, many forecasting exercises nowadays incorporate 'off-model' information such as assumption about future interest rate. To mention a few, the official reports from U.S greenbook forecast, Riksbank and Bank of England all contain simulations conditioned on either a constant future interest rate, or a rate path inferred from market expectation. All these cannot be done with the usual shock analysis and conventional forecasting exercise.

Stress testing is another good example to illustrate the importance of scenario analysis. By testing the vulnerabilities of financial institutions under hypothetical stressed scenarios, policymakers could be forewarned against potential risk and take precautionary move to avoid undesirable consequences. Because of this, stress testing has become a regulatory requirement for the banking industry in the U.S. and some other jurisdictions after the global financial crisis in 2008.

Scenario study can be a powerful technique that facilitates policy analysis, yet it is not widely applied in current macroeconomic studies. This is possibly due to the reason that modern macroeconomic research relies extensively on complex dynamic models but lacks of a generic and efficient algorithm that can generate model-consistent scenario analysis results. Technically speaking, such analysis requires the analyst to be able to simulate the economy while respecting restrictions imposed on some endogenous observables at a future time. Sampling of a dynamic model under a future scenario immediately faces analytical intractability of the conditional distribution/density function of the target model even for high-dimensional linear Gaussian systems, and becomes particularly challenging with the added parameter uncertainty. This makes direct sampling from the target conditional density a difficult task if not impossible.

A number of research articles have attempted to devise such sampling algorithm, but all of them face some kinds of limitation. The literature is known as conditional forecasting. Waggoner and Zha (1999) provides the earliest solution for conditional forecasts under VAR models while considering parameter uncertainty. According to Jarociński (2010) and Bańbura et al. (2015), this algorithm is not computationally efficient, and can easily become impractical or infeasible for high dimensional data. Subsequent research such as Bańbura et al. (2015) proposes the use of Kalman filter technique as a simulation smoother, and Maih (2010) proposes an approach that can handle soft ending conditions. According to Maih (2010), all Kalman filter based algorithms can only deal with hard ending conditions, i.e, a fixed future value but not a set of values. All the available approaches share one commonality; that is, they rely on solving the exact conditional distribution of a complex system in some analytical form. Their applicability is thus restricted to linear Gaussian systems, and the derivation task is usually quite involved and model-specific. In fact, until today, there exists no generic algorithm that can be easily extended to non-linear and/or non-Gaussian models.

In this paper, we propose a generic and efficient sampling method for dynamic scenario analyses. This method relies on the bridge sampling technique of Duan and Zhang (2016), which is built on an explicit recognition that the sampling difficulty does not reside with the joint distribution/density function of the system, but rather with the complexity of the probability/density for the conditioning event. However, the latter can be treated as an irrelevant norming constant when an importance sampling scheme is deployed. The basic idea behind importance sampling is that sampling from the target distribution is equivalent to sampling from a different proposal distribution with the correct importance weights. Such scheme in general consist of three steps: (1) drawing initial particles from a proposal density, (2) assigning them the correct importance weights and (3) resampling according to the weights.

In our scenario analysis context, each particle is a set of bridge paths for observables and our target distribution is their conditional density given the scenario-defined condition. The proposal density is associated with a generating model that comprises two parts – (1) a convenient Gaussian bridge to approximate the driving process defining the future scenario, say, inflation rate process, and (2) a companying Gaussian system approximately describing the remainder (output, employment, interest rate, etc.) conditional on the driving process. The generating model can be easily sampled, and the final set of sample paths under the target model and meeting the condition defined by the future scenario can be obtained by closing the gap between the target and generating models sequentially with a density-tempering technique without having to know the conditional probability/density of the defining scenario, i.e., an irrelevant norming constant. We further extend this method to a general state-space setting with latent variables and account for parameter uncertainty by allowing for Bayesian updating on the parameter values influenced by the occurrence of future scenario. It is also flexible in accommodating both hard and soft ending conditions and can easily deal with degenerate state-space models.

The flexibility of our bridge sampling method enables a scenario-based study of different models. There has been continuous debate on the relative forecasting performance between reduced-form econometric models like vector autoregression (VAR) and more theory-based structural models such as dynamic stochastic general equilibrium models (DSGE). On the one hand, VAR models especially unrestricted ones, are said to be most flexible in matching historical data and hence should deliver more accurate forecast if the interrelationship among model variables remain stable. On the other hand, DSGE models incorporate forward looking behavior of economic agents and are said to provide more interpretable and accurate prediction on shock responses if the decision-making processes are modeled correctly. In Smets and Wouters (2007), the authors show that their 14-variable DSGE model produces comparable marginal data likelihood and forecast performance as a 4^{th} order Bayesian VAR model and an unrestricted VAR(1) model using seven U.S. macro data series (GDP, consumption, investment, labor, wage, inflation and interest rate) from 1966:Q1 to 2004:Q4.

In our empirical analyses, we first re-examine the conventional forecast performance of the three models using extended US data from 1966:Q1 to 2017:Q4.

We find that the DSGE model continues its superior performance during the Great Recession (December 2007 - June 2009) but loses out to the BVAR(4) model in subsequent recovery/expansion period. After that, we perform a scenario-based study of these three competing models, focusing on investigating how they perform differently under inflation targeting and interest rate scenarios. Since dynamic scenario analysis is much more demanding on the model dynamics, it is conceivable that the outcomes of the three models will be different. Indeed, our empirical results suggest that under the same inflation target, predicted paths for the remaining six observables, computed at the end of the data sample, i.e., 2017:Q4, vary across models. For example, the predicted growth rate of real wage from the DSGE model is about twice of the rate predicted by the other two VAR models throughout the four quarters in 2018.

In reaction to the bursting of the Dot-com bubble, the US Federal Reserve lowered the policy rate 11 times in 2001 from 6% to around 2%. As expected, the unconditional forecast under all three models fails to predict such drastic rate cuts, and by extension also fails to forecast other macro variables. Conditional forecasts of the three models based on the realized interest rate of 2.1% at the end of 2001, however, perform quite differently. Both VAR(1) and BVAR(4) models produce conditional forecasts on other macro variables matching closer to their actual realizations. Surprisingly, the prediction from the DSGE model becomes even worse with conditioning on the realized interest rate, supporting our contention that scenario analysis may serve as a powerful way of discriminating competing models.

2 Macroeconomic models and scenario analysis

Dynamic macroeconomic models are generally classified as reduced-form econometric models, like VAR, or structural models such as DSGE. Both VAR and DSGE models have been widely used to perform forecasting and structural analysis. There are a number of papers addressing the differences between these two approaches, and comparing their unconditional forecasting performance. The general conclusion is that forecasts from DSGE models are not more accurate than times series models, but neither are they any worse. Wickens (2014) argues that this is due to the dynamic structure in the solution of DSGE models. Their backward-looking dynamics gives them a forecasting performance similar to time series models and their forward-looking dynamics, which consists of expected value of future endogenous variables, is difficult to forecast accurately.

Dynamic scenario analysis is much more demanding on the model dynamics. Hence, whether the conclusion for conventional forecast performance can still hold under scenario analysis is an interesting question. In contrast with the large literature on conventional forecast studies, little attention has been paid to macro scenario analyses due to the technical complexity of sampling conditional bridge paths. For a simple process like AR(1) with Gaussian innovations, the exact bridge distribution can be easily derived. But with a larger-dimensional VAR or the DSGE model, the exact solution may become analytically messy. If the model in question is nonlinear or non-Gaussian, the exact solution becomes analytically intractable.

Hence, developing a versatile, easy-to-implement and reliable bridge-sampling method will be critical to dynamic macro scenario analyses.

In this section, we first describe the general econometric framework for both VAR and DSGE models. Then we show how they can be cast into the linear state-space form comprising (1) a VAR(1) process driving the state variables and (2) a measurement equation linking the state variables to the observable measures. After that, we present our bridge sampling method under the state-space setting. To facilitate our discussion, we use inflation targeting as an example of conditioning scenario, and illustrate our algorithm based on the state-space representation of the VAR(1) model in Smets and Wouters (2007). Subsequently, we highlight the robustness of our method in dealing with soft conditions and degenerate state-space models such as the DSGE model of Smets and Wouters (2007). Finally, we make comparison to an alternative method, i.e., rejection sampling, when it is possible.

2.1 VAR, DSGE and their linear state-space representation

A general n-dimensional VAR(p) model is of the form:

$$\begin{aligned} \boldsymbol{y}_t &= \boldsymbol{A} + \sum_{l=1}^{p} \boldsymbol{B}_l \boldsymbol{y}_{t-l} + \boldsymbol{e}_t \\ \boldsymbol{e}_t &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}^e) \end{aligned} \tag{1}$$

where $\boldsymbol{y}_t = (y_{1,t}, y_{2,t}, \ldots, y_{n,t})$ denotes the vector of observations at time t. $\boldsymbol{A}$ is the constant term if any. $\boldsymbol{B}_l$, $l = 1, 2, \ldots, p$ are the autoregressive matrices, each with a dimension $n \times n$, and the residual $\boldsymbol{e}_t$ follows a white noise process with mean $\boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma}^e$.

Since the publication of Sims (1980), VAR models have been widely used to provide empirical benchmark for structural models, in both forecasting exercise and structural analysis. However, as pointed out by Bańbura et al. (2015) and many other papers, the availability of huge data sets nowadays raises a trade-off between the excessive simplicity of the models, leading to mis-specification due to omitted variables, and their excessive complexity, with many free parameters leading to a large estimation uncertainty. To be specific, in the above n-dimensional VAR(p) model, without any restrictions on the parameters, we have $n \times (n \times p + 1$ parameters from vector $\boldsymbol{A}$ and autoregressive matrices $\boldsymbol{B}_l, l = 1, 2, \ldots, p$, and $\frac{n\times(1+n)}{2}$ parameters from covariance matrix $\boldsymbol{\Sigma}$ to be estimated. This leads to unreliable estimation and inference if either the dimension or the order of lags is large comparing to the data length. This is commonly referred to as the 'curse of dimensionality' for VAR models. To mitigate this problem, the Bayesian approach has been introduced to combine information from the data and some prior belief on model parameters. It results in the estimates shrinking towards their prior expectations, which usually come from economic theories or previous empirical results.

(B)VAR models can be straightforwardly converted to state-space formulation. For first order (B)VAR models, the auto-regressive process itself provides the state transition equation. And the state variables is identical to the observables because there is no latency in the system. That is:

$$\begin{aligned} \boldsymbol{s}_t &= \boldsymbol{A} + \boldsymbol{B}_1 \boldsymbol{s}_{t-1} + \boldsymbol{e}_t \\ \boldsymbol{y}_t &= \boldsymbol{s}_t \\ \boldsymbol{e}_t &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}^e). \end{aligned} \tag{2}$$

For a general p-th order (B)VAR model given in (1), we first write the model into VAR(1) form:

$$\begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 & \dots & \dots & \boldsymbol{B}_p \\ \boldsymbol{I} & \boldsymbol{0} & \dots & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{t-1} \\ \boldsymbol{y}_{t-2} \\ \vdots \\ \boldsymbol{y}_{t-p} \end{bmatrix} + \begin{bmatrix} \boldsymbol{e}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}. \tag{3}$$

By denoting:

$$\boldsymbol{s}_t = \begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p+1} \end{bmatrix}, \quad \tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}, \quad \tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 & \dots & \dots & \boldsymbol{B}_p \\ \boldsymbol{I} & \boldsymbol{0} & \dots & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \quad \tilde{\boldsymbol{e}}_t = \begin{bmatrix} \boldsymbol{e}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix},$$

the state-space representation for a n-dimensional (B)VAR(p) model can be written as:

$$\begin{aligned} \boldsymbol{s}_t &= \tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}} \boldsymbol{s}_{t-1} + \tilde{\boldsymbol{e}}_t \\ \boldsymbol{y}_t &= [\boldsymbol{I} \quad \boldsymbol{0} \quad \dots \quad 0]\, \boldsymbol{s}_t \\ \tilde{\boldsymbol{e}}_t &\sim N\left(\boldsymbol{0}, \begin{bmatrix} \boldsymbol{\Sigma}^e & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \end{bmatrix}\right) \end{aligned}$$

In the above, $\boldsymbol{I}$ is the n-dimensional identity matrix. Note that for the linear state-space model converted from a (B)VAR process, it is always degenerate due to the absence of measurement error. In other words, the state variable $\boldsymbol{s}_t$ contains no more information than $\boldsymbol{y}_t$, and its lags is converted from a higher-order (B)VAR.

Both VAR and BVAR models are unavoidably subject to 'Lucas critique', which argues that such econometric models should not be used for structural analysis as it has no economic theory behind, and past data cannot be representative of future reactions to new shocks. Such criticism motivated the development of structural models with micro foundations. DSGE is one of them, and it has become a very popular approach in macro analysis. In particular, the DSGE model developed by Smets and Wouters (2007), which was originally used to explain US macro fluctuation, has become the benchmark model for many subsequent structural analyses.

There are a few ways to solve DSGE models, one commonly used approach involves log-linearization of the model and rewrite it into the canonical linear rational expectation form:

$$\begin{aligned} \boldsymbol{G}_0 \boldsymbol{s}_t =& \boldsymbol{G}_1 \boldsymbol{s}_{t-1} + \boldsymbol{D}_1 + \boldsymbol{\Omega}_1 \boldsymbol{z}_t + \boldsymbol{\Omega}_2 \boldsymbol{\eta}_t \\ \boldsymbol{z}_t \sim& N(\boldsymbol{0}, \boldsymbol{\Sigma}^z). \end{aligned}$$

where $\boldsymbol{s}_t$ denotes the state variables, which includes all endogenous variables and artificial variables that are necessary to make the linearized model a VAR(1) representation. $\boldsymbol{z}_t$ denotes the exogenous variables (shocks) with mean zero and covariance $\boldsymbol{\Sigma}^z$, and $\boldsymbol{\eta}_t$ is endogenously determined one-step-ahead expectation errors. Solving the model according to Sims (2002) gives rise to the following solution:

$$\begin{aligned} \boldsymbol{s}_t =& \boldsymbol{G} \boldsymbol{s}_{t-1} + \boldsymbol{D} + \boldsymbol{\Omega} \boldsymbol{z}_t \\ \boldsymbol{z}_t \sim& N(\boldsymbol{0}, \boldsymbol{\Sigma}^z) \end{aligned} \tag{4}$$

with matrix $\boldsymbol{G}$ and $\boldsymbol{\Omega}$ being functions of the original structural parameters to provide identification. As seen from the above equation, the dynamic of the DSGE model is essentially a VAR(1) process with latency. The observable measures are linked to state variables $\boldsymbol{s}_t$ through a measurement equation:

$$\begin{aligned} \boldsymbol{y}_t =& \boldsymbol{F} + \boldsymbol{M} \boldsymbol{s}_t + \boldsymbol{v}_t \\ \boldsymbol{v}_t \sim& N(\boldsymbol{0}, \boldsymbol{\Sigma}^v) \end{aligned} \tag{5}$$

where $\boldsymbol{F}$ is the constant term, $\boldsymbol{M}$ is the selection matrix and $\boldsymbol{v}_t$ represents the measurement error if any. The forms of parameter matrices and vectors are specific to a DSGE model such as Smets and Wouters (2007).

2.2 Inflation targeting – an example

Price stability is the most well-recognized policy objective around the world with more than 28 countries adopting an explicit inflation target. Here we take the 2% annual inflation target set by the US as an example, and demonstrate the idea of our bridge sampling method based on the VAR(1) model of Smets and Wouters (2007). Implementation details for general state-space models are provided in Appendix A.

The state-space representation of the VAR(1) model of Smets and Wouters (2007) is given by equations in (2), with $\boldsymbol{y}_t$ being the seven-dimensional observable variable:

$$\boldsymbol{y}_t = \begin{bmatrix} dlGDP_t \\ dlCONS_t \\ dlINV_t \\ dlWAG_t \\ lHOURS_t \\ dlP_t \\ FEDFUNDS_t \end{bmatrix} \tag{6}$$

where l and ld stand for 100 times log and log difference, respectively. dlP_t, $lHOURS_t$ and $FEDFUNDS_t$ correspond to the inflation, labor input and the federal funds rate. Inflation is calculated as the log difference of the GDP deflator and labor input is measured by log hours worked (demeaned). $dlGDP_t$, $dlCONS_t$, $dlINV_t$ and $dlWAG_t$ denote the log differences in output, consumption, investment and wage. All observed series are in quarterly frequency.

Suppose at time T, we are interested to know how the US economy would look like conditional on achieving its annual inflation target of 2%. Given the data frequency and definition in Smets and Wouters (2007), 2% annual inflation is equivalent to $\sum_{t=T+1}^{T+4} dlP_t = 2$. Hence we want to simulate the model by generating sample bridge paths subject to that condition. In order to make use of the notations and formulas in Appendix A, we cast the VAR(1) model in our general state-space formulation in equations (7)-(8) with $\boldsymbol{C} = \boldsymbol{A}$, $\boldsymbol{B} = \boldsymbol{B}_1$, $\boldsymbol{\Omega} = \boldsymbol{I}$, $\boldsymbol{\epsilon}_t^s = \boldsymbol{e}_t$, $\boldsymbol{\Sigma}^s = \boldsymbol{\Sigma}^e$, $\boldsymbol{\mu} = \boldsymbol{0}$, $\boldsymbol{M} = \boldsymbol{I}$, and zero measurement error.

Our bridge-sampling method relies on that of Duan and Zhang (2016), where the key is to use the Sequential Monte Carlo (SMC) method. SMC refines upon an importance sampling algorithm by tempering the importance weights in a sequential manner. This is to mitigate the sample impoverishment issue caused by uneven weights. Furthermore, to reduce duplicated particles after resampling, a support boosting step is implemented to restore sample diversity. Our method adapts that of Duan and Zhang (2016) to a general state-space setting and adds parameter uncertainty that is expected in real-world applications. Parameter uncertainty is handled by augmenting the space describing the bridge paths of observables with the model parameter space. To adapt the method to a general state-space setting, we further extend the space to include paths for unobserved state variables because they form part of the information set needed for advancing the model forward. Notice that by the transition equation, the state variables and its innovation terms are equally informative once the previous state is known. Hence in the augmented system, we can substitute the paths for unobserved states by the paths for its innovation term plus the initial state at T. We show in Appendix A.5 that this substitution makes it more convenient in dealing with degenerate models. In summary, our complete augmented space includes (1) the bridge paths for the observable variables, (2) the model parameters, (3) the initial state variables at T, and (4) the paths for shocks from $T+1$ onward.

Algorithmically, we start with an initialization sampler that is associated with a 'wrong' approximating model but easy to sample from. In our VAR(1) example, the initialization for model parameters comes from the frequentist asymptotic distribution. And the initial state $\boldsymbol{s}_T$ simply equals to $\boldsymbol{y}_T$. As for the bridge paths for the observables and shocks, their initialization consists of two steps. In the first step, we assume that the conditioning variable, which is dlP_t in our inflation targeting example, follows a Gaussian AR(1) process while knowing it is untrue. Then the bridge distribution conditional on the scenario $\sum_{t=T+1}^{T+4} dlP_t = 2\%$ can be analytically derived.[1] In the second step, the paths for the remaining six macro variables can be sampled with a companion system obtained by a simple decomposition given by equations (12)-(14) in Appendix A.1 while the paths for shocks are sampled according to equations (15)-(16).

[1]The derivation is similar to how we derive equation (11) in Appendix A.1. The analytical formula is available upon request.

Following the idea of importance sampling, the points from our initialization sampler are legitimate when coupled with the correct importance weights, which is the ratio of their density under the target model over that under the initialization sampler. (See equation (21) in Appendix A.2) These two densities can be evaluated with equations (9) and (20). The key to the method is to recognize that the last term in the target density formula, i.e., the probability of hitting the inflation target, is actually an irrelevant constant as far as importance sampling is concerned. Moreover, the density of the initial state also disappears from the likelihood ratio through cancellation.

The importance weights are likely to be uneven across the sampled points due to the difference between the target model and the generator. Therefore, density tempering the likelihood ratio in a sequential manner will be needed, and resampling at each intermediate stage must also be conducted to even out the importance weights so as to prevent sample impoverishment. The tempering sequence is chosen adaptively to ensure a certain variety of points being resampled. After each resampling step, one also needs to boost the empirical support through performing the Metropolis-Hastings (MH) moves to restore sample diversity. Implementation details for density tempering and support boosting are provided in Appendix A.2-A.3.

It is worth noting that the above bridge-sampling method can be easily implemented if the parameter is treated as known, for example, the typical practice of deploying the point estimate for the parameter. Algorithmically, one simply skips the step of sampling parameter in both initialization and support boosting. The likelihood ratio formula becomes even simpler, with the densities of model parameters dropped out and all other terms evaluated at the fixed parameter value.

While simulating the economy under this 2% inflation target, some may take a shortcut by assuming a constant path for inflation. However, as criticised by Waggoner and Zha (1999), that means the conditioning variable is assumed to stop responding to the state of the economy and become exogenous at each and every forecast date, which may violate the dynamic of the given model. Moreover, policy objectives are rarely specified as a fixed path, instead, they are usually given as a target level or range over a particular time period. For instance, according to the Bank of England, a target of 2% does not mean that inflation will be held at this rate constantly. That would be neither possible nor desirable. Instead, the Monetary Policy Committee's aim is to set interest rates so that inflation can be brought back to target within a reasonable time period, say one year, without creating undue instability in the economy. Note that our method only restricts the annual inflation to be 2% while allowing the paths in-between to be flexible while respecting the model dynamics. Therefore, we are able to address policy objectives in a more natural and logically consistent way that better mimics the reality.

2.3 Dealing with soft conditions and degenerate linear state space models

In real-life applications, many policy objectives or stressed scenarios are specified as a range or interval instead of a single value. One recent example is that India announced their inflation target to be 4% with a tolerance interval of $\pm 2\%$ for the next five years until 2021. This type of range scenario is defined as soft condition in Waggoner and Zha (1999). The simulation algorithm proposed in their paper that deals with soft conditions is essentially a simple rejection method with some variance reduction technique. It is highly inefficient as it generates many unsatisfactory paths that are rejected eventually.

In contrast, our bridge-sampling method is flexible and can accommodate soft conditions with just one additional sampling step. Recall that in the initialization, we need to draw bridge paths for the conditioning variable, i.e. dlP_t in this example, based on its estimated AR(1) process. The bridge process depends on the value of the scenario constraint's expression, which in this case is the sum of quarterly inflation in the subsequent four periods. In the case of hard condition discussed above, $\sum_{t=T+1}^{T+4} dlP_t = 2\%$ is the only permissible value. With a targeted range of $4 \pm 2\%$ as in the case of India, it implies $\sum_{t=T+1}^{T+4} dlP_t \in [2\%, 6\%]$. Before drawing the bridge paths for inflation, one simply samples the sum $\sum_{t=T+1}^{T+4} dlP_t$ according to a truncated normal distribution defined by the target range.

Degeneracy typically requires more care due to the presence of the Dirac delta function in the density formula. However we show in Appendix A.5 that this does not cause any numerical difficulty to our bridge sampling method. There are two types of degeneracy that could arise in the general state-space formulation in equations (7)-(8). One is caused by zero measurement error, i.e., observable $\boldsymbol{y}_t$ is deterministic given $\boldsymbol{s}_t$. This is always the case for (B)VAR models because they assume no latent states. The other type of degeneracy is associated with dimension mismatch between the state variable $\boldsymbol{s}_t$ and the its driving shocks, $\boldsymbol{\epsilon}_{s,t}$. This is quite common in state-space representations converted from DSGE models including Smets and Wouters (2007). This model has 14 endogenous variables driving by seven exogenous shocks. To express it as an autoregressive process of order one, we need to translate it into a 50-state variable VAR(1) transition system with the innovation $\boldsymbol{\epsilon}_{s,t}$ being seven-dimensional. In addition, the DSGE of Smets and Wouters (2007) assumes no measurement error. Thus it contains both types of degeneracy, and we use it to illustrate in details how to deal with degeneracy in Appendix A.5.

2.4 Rejection Sampling

Rejection sampling is an alternative approach to scenario analysis when the scenario is defined by a soft ending condition, and hence can be used to provide a quality check on our bridge sampling method. When there is no parameter uncertainty and no latent state, it is easily implemented by simulating the model forward in the unconditional manner and reject those paths that fail to meet the scenario condition. However, when the conditioning band is narrow, this rejection method will become highly inefficient as it generates many unsatisfactory paths that are rejected. If the conditioning band contains a single point, i.e., hard conditions, rejection sampling will be infeasible as the probability of hitting a fixed ending point is essentially zero. Furthermore, with parameter uncertainty and under state-space setting, rejection sampling becomes much more complicated. It involves drawing model parameters from the conditional distribution given the scenario condition. And when initial state is not fixed, it also requires sampling $\boldsymbol{s}_T$ from the smoothed distribution conditional on historical observations, scenario condition and drawn parameter values.

3 Applications

In this section, we demonstrate how dynamic scenario analysis can provide insights for policy makers under a well-recognized macro objective. The analysis is applied to three models: the unrestricted VAR(1), BVAR(4) and DSGE models from Smets and Wouters (2007). We show that models with comparable forecasting power can provide qualitatively different inferences under the same conditioning scenario. We also provide performance comparison of the three models on scenario analysis versus unconditional forecast, as well as statistical properties using root mean squared error (RMSE) over various time spans.

3.1 Data, estimation and quality check

Our dataset consists of the same seven U.S. macro variables as in Smets and Wouters (2007), which include: log difference of output, consumption, investment and wage, log hours worked (demeaned), inflation and federal funds rate. All series are in quarterly frequency and extended to cover the period from 1966:Q1 to 2017:Q4. For detailed data definition, construction and sources of data, one may refer to Herbst and Schorfheide (2015).

The estimation for the unrestricted VAR(1) and BVAR(4) follows that of Smets and Wouters (2007) exactly, whereas for the DSGE model we adopt the Sequential Monte Carlo approach proposed by Herbst and Schorfheide (2014).[1] The DSGE model estimation results are presented in Appendix B, and we compare them with the posterior distribution documented in Smets and Wouters (2007) which used data before 2005. We find that the standard errors for risk premium, exogenous spending, investment, and monetary policy shocks are lower than the estimates using data before 2005, whereas that for wage mark-up shock is substantially higher. In addition, the productivity, risk premium and investment shocks are more persistent than before. As for the structural parameters, the cost of capital adjustment falls but the cost of capital utilization adjustment cost rises. There is less habit formation after 2005, and the degree of both price and wage stickiness increases. Furthermore, the steady state growth rate is lower than before.

[1]For prior implementation for the BVAR model, linearized DSGE model equations and prior distributions, see Smets and Wouters (2007).

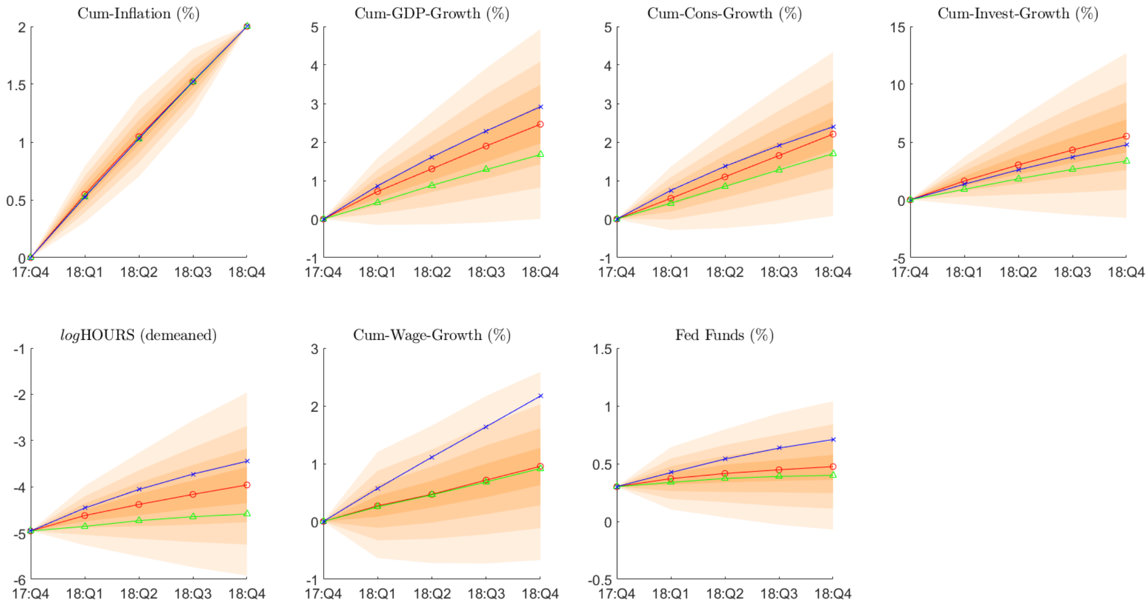
Before demonstrating the use of our bridge sampling method on a real policy scenario, we first ascertain the quality of the technique by comparing it with rejection sampling. To avoid the complexity in dealing with parameter uncertainty in the rejection method, our quality check is performed with fixed parameter values. For the DSGE model, we also assume the initial state is fixed by using the mean value of the filtering distribution. We apply both sampling methods to the scenario that requires the inflation in 2018 to be within the range between 5% and 5.5%. It is worth noting that more than 99% of the generated paths under rejection sampling were tossed away because they failed to meet the scenario condition. Figures 5 and 6 in Appendix C present the quantile-to-quantile (QQ) plots of the two empirical distributions (bridge sampling vs simple rejection sampling) from the VAR(1) and DSGE models, based on 10,000 paths. We show the results for all seven observables at three different future time points. The results reveal that the paths drawn from the bridge sampler match well with those generated by the rejection sampler. To conserve space, we omit the QQ-plots for the BVAR(4) model.

3.2 Application on inflation targeting

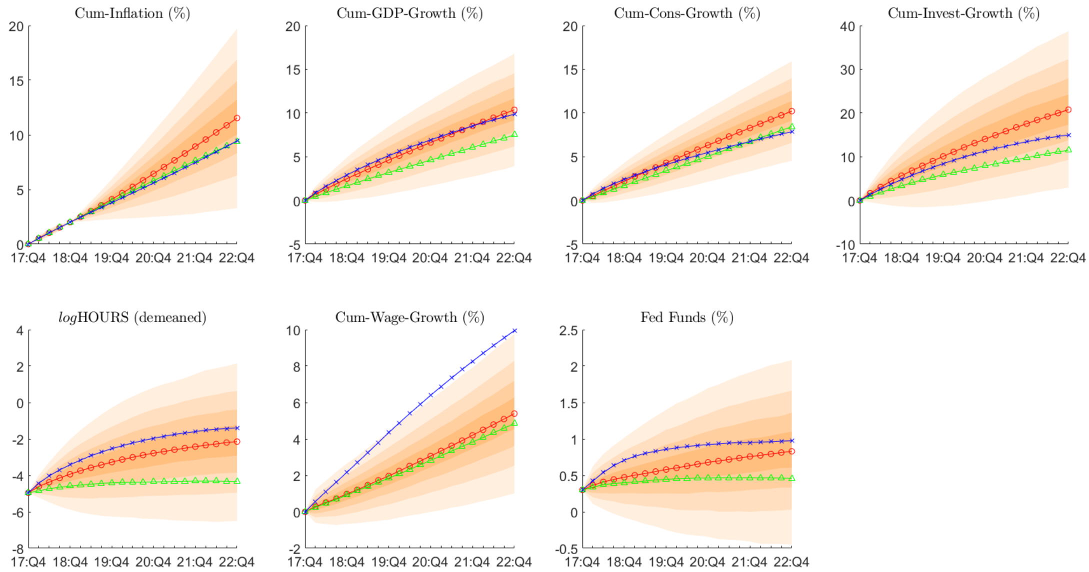
Inflation targeting is a monetary policy framework pioneered in New Zealand in 1990, with currently more than 28 countries adopting it. In order to ensure central banks' accountability, policy authorities establish explicit quantitative inflation targets for a specific period ahead. As an illustration, using the models estimated in section 3.1, we perform scenario analysis to see how the US economy evolves under the condition that inflation in 2018 hitting the 2% official target. We present in Figure 1 the empirical results from all three models in one combined graph for easy comparison.

The upper panel of Figure 1 plots the predicted paths for the seven observables over 2018. It suggests that even under the same annual inflation rate target, predicted trends on major economic indicators vary across models. In general, the BVAR model is the most conservative in predicting growth while the DSGE is the most optimistic except for investment. For instance, both the VAR(1) and DSGE models predict higher GDP growth than the BVAR model, with the predicted rate from the DSGE model being nearly twice of that from the BVAR model . Similarly for the cumulative growth of real wage, the predicted rate from the DSGE model is also about twice of those from the two VAR models. Furthermore, its predicted path falls out of the 80% confidence band of the VAR(1) model starting from the third quarter of 2018. As for the monetary policy, the DSGE model suggests that the US Federal Reserve under the 2% inflation target would need to raise the annualized federal funds rate to 2.7% by the end of 2018, while both the VAR(1) and BVAR(4) models suggest a less aggressive move. Note that the federal funds rates in the figure are quarterly rates.

The differences are more evident when we generate the 5-year simulation conditional on the first year's inflation rate hitting the 2% target. Panel (b) of Figure 1 shows that even the very small differences in the predicted investment growth in 2018 leads to considerable differences on the investment growth over a five-year horizon. The discrepancies in predicted labor input and real wage



(a) 1-year simulation while targeting the first year's inflation at 2%



(b) 5-year simulation while targeting the first year's inflation at 2%

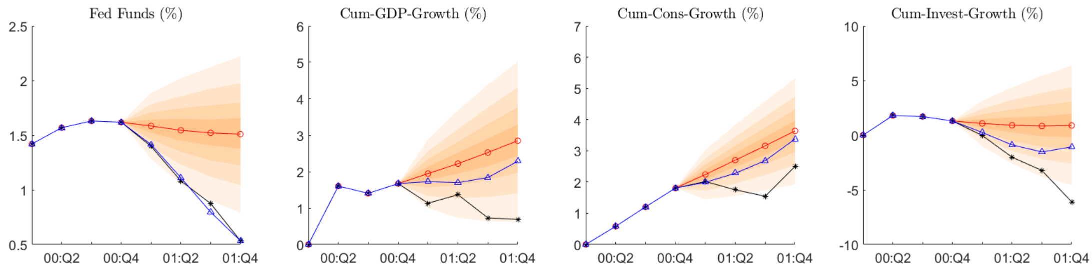
Figure 1: Scenario analysis with an inflation target

The graph is based on $N = 10,000$ bridge sample paths generated for each model. The red solid line with circles represents the mean paths for the VAR(1) model, with the orange shaded areas represent 10 to 90 percentile of the conditional distribution of the VAR(1) model. The green line with triangles and blue line with crosses represent the mean paths for the BVAR(4) and DSGE models, respectively. All growth rates are cumulative in both Panels (a) and (b). The labor input is log hours worked (demeaned), and the federal funds rate is a quarterly rate.

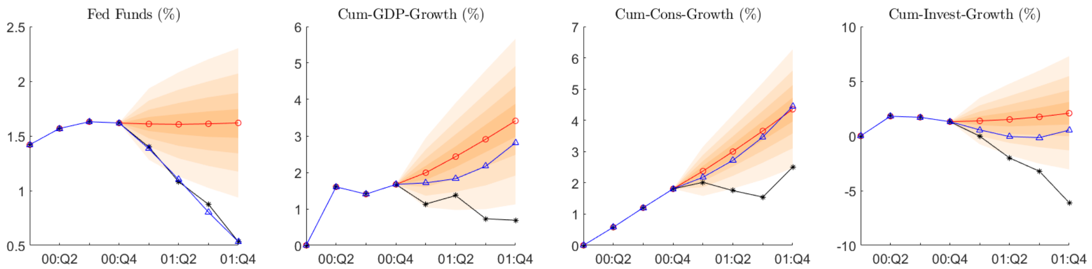
growth among the three models are more significant with the wage growth predicted by the DSGE model to surge above the 90 percentile of the distribution based on the VAR(1) model.

3.3 Dynamic scenario analysis vs forecast performance

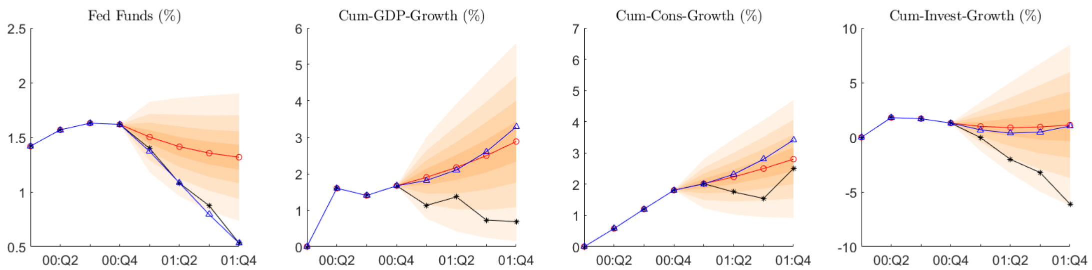
Policy authorities usually have valuable 'off-model' information such as knowledge about future policy target rate. Forecast conditioned on such 'insider' information is supposed to improve forecast accuracy if the model correctly characterizes the relationship of the conditioning and other model variables. In this subsection, we present a case study where the actual implemented federal funds target rate is far away from the level predicted by the unconditional forecast. We examine how a scenario analysis would have provided different insights on the forecast of the economy in contrast with the unconditional analysis.



(a) VAR(1) model



(b) BVAR(4) model



(c) DSGE model

Figure 2: Scenario analysis vs unconditional forecast

The black solid line with asterisks represents the realized data. The red line with circle show the mean paths from unconditional forecast, with the orange shaded area represents 10 to 90 percentile of the unconditional prediction distribution of the respective model. The blue line with triangle represents the mean paths generated under scenario analysis.

With the bursting of the Dot-com bubble, the slowdown of the US economy in late 2000 became apparent and intensified in the first half of 2001. To fight the recession, the US Federal Reserve lowered the policy rate 11 times in 2001 from higher than 6% to around 2%. It is almost certain that conventional unconditional forecast will fail to predict such drastic and intensive rate cut due to the high interest rate persistence estimated from historical data. This is confirmed with Figure 2 above, which shows all three models predicting a much gentler rate decline to around 5.2-6% at the end of 2001. This leads to unreliable forecast for other variables as well. By looking at the results of unconditional forecast, we see that the DSGE model of Smets and Wouters (2007) delivers comparable out-of-sample forecast with the two VAR models, which is consistent with the statistical evidence provided in Smets and Wouters (2007). But overall, all three models perform poorly with the realized paths for output, consumption and investment barely touch the 10 percentile bound of the unconditional forecast.

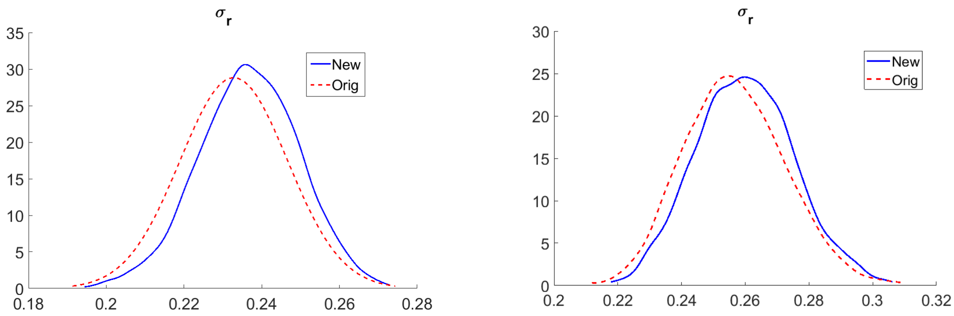
In such circumstances, scenario analysis provides more valuable information and addresses policy makers' needs directly. It allows policy makers to (1) incorporate advanced information on future rates to produce more informative forecast result and (2) experiment with different scenarios to identify the optimal strategy. As an illustration, we perform a scenario analysis constrained by the interest rate hitting the realized 2.1% at the end of 2001, and the result is also presented in Figure 2. As one can expect, the predictions from scenario analysis for the VAR and BVAR models match closer with the actual data. Surprisingly however, the predictions for GDP and consumption from the DSGE model become even worse after conditioning on the realized interest rate. This lends further support to our argument that the (B)VAR and DSGE models provide considerably different predictions under the same conditioning scenario. Therefore, comparable forecasting performance does not necessarily imply similar scenario analysis results.

3.4 Scenario-induced impact on parameter uncertainty

In real-life applications, the belief on model parameters is continuously updated upon the arrival of new information. Therefore, the occurrence of a future scenario should cause a revision to parameter estimates in order to stay consistent with the contemplated scenario, a point already made by Waggoner and Zha (1999). Hence, the conditional forecast for variables of interest, especially when the conditioning variables take on extreme values, should also reflect the change in the parameter distribution caused by the scenario. Our bridge sampling method has naturally incorporated a scenario's impact on parameter uncertainty.

To appreciate the effect of ignoring a scenario's impact on parameter values, we compare the scenario-impacted sampling distribution with the one derived from the data sample for the VAR(1) and DSGE models using data up to the end of 2000, treating it as the time point for conducting the scenario analysis. In order to see a clearer effect on parameter uncertainty, we generate conditional paths over the year 2001, with a more extreme artificial scenario that the federal funds rate drops to 0% at the end of 2001. Figure 3 below demonstrates the effect on some model parameters for the VAR(1) and DSGE models. The left panel shows a rightward shift on the scenario-impacted sampling distribution for the standard deviation of the innovation term associated with the interest rate equation in the VAR(1) model, whereas the original one is the asymptotic distribution for the OLS estimate with data up to the end of 2000. And the right panel also suggests a scenario-induced rightward shift on the estimate for the standard error of monetary policy shock parameter, i.e., σ_r, in the DSGE model of Smets and

Wouters (2007), where the original distribution is its Bayesian posterior up to the end of 2000.



(a) VAR(1) parameter

(b) DSGE parameter

Figure 3: Effects on parameter estimates

Panel (a) shows the distribution of the standard deviation of the interest rate innovation term in the VAR(1) model, whereas panel (b) displays the distribution of the parameter σ_r in the DSGE model of Smets and Wouters (2007). The red dashed line represent respectively the OLS estimator's asymptotic distribution for the VAR(1) model and the posterior distribution for the DSGE model using data up to 2000:Q4. The blue curves are the scenario-induced sampling distributions after factoring in the conditioning future scenario for the federal funds rate to hit zero at the end of 2001. Except for the asymptotic normal distribution for the OLS estimator, the other three distributions are kernel density functions using their respective SMC samples.

3.5 Statistical evidence on scenario-based performance analysis

Before we discuss the scenario-based performance of the three models, we document their unconditional forecast performance. Smets and Wouters (2007) concludes that the DSGE model is superior than the other two VAR models in terms of out-of-sample forecast by showing the root-mean-square-error (RMSE) over the period 1990:Q1 - 2004:Q4. Note that the earlier period of data from 1966:Q1 to 1989:Q4 was used for initial estimation of the models. We re-examine this conclusion by focusing on the out-of-sample period from 2005:Q1 to 2017:Q4 while using the 1966:Q1 - 2004:Q4 period for initial estimation of the models. The out-of-sample performance results are presented in Appendix D. It worth mentioning that following Smets and Wouters (2007), forecast errors are calculated in terms of GDP, consumption, investment and wage levels instead of quarterly growth rate when comparing their forecast performance. We note from Table 8 that in the recent decade, the BVAR(4) model has overtaken the DSGE model to become the best predicting model for GDP, labor input, inflation and interest rate at most forecast horizons. The DSGE model, which used to deliver the best prediction for all variables, now only performs better in forecasting consumption.

To further investigate which sub-period causes the deteriorated performance of the DSGE model, we divide the forecast period into two sub-periods. According to the NBER Business Cycle Dating Committee, the last expansion (after Global Financial Crisis) began in June 2009. Hence we use 2009:Q2 as a natural break point, and present the forecast performance before that in Table 9, and after that in Table 10 in Appendix D. Hardly surprising, the prediction errors are much larger for the period before 2009:Q2 as it contains the Great Recession period during which economic indicators are highly volatile and hard to predict. It is interesting to note that the DSGE model continued its good performance during the crisis time, but lose out to the BVAR model almost completely in the subsequent expansion period.

In order to study the scenario-based performance of a model, we can run the same back-testing exercise with scenario constraints. In other words, we perform a sequence of scenario analyses with expanding data windows over some forecast period. In Table 1, we perform back-testing conditional on the realized federal funds rate, for the three models over the period 2005:Q1 - 2017:Q4. Again, the performance on individual variables is measured by root mean squared error (RMSE). We also compare this result to that of unconditional forecast given in Table 8, with the percentage gains or losses in RMSE presented in Table 2. Note that a higher (positive) gain corresponds to a lower RMSE for scenario analysis as compared to unconditional forecast.

Table 1: Back testing on scenario analysis using realized federal funds rates (2005:Q1 - 2017:Q4)

Forecast horizon	GDP	CONS	INV	Wage	*l*Hours	*dl*P	Average	Overall (SW)*
				VAR(1)				
1q	0.63	0.77	1.83	**1.07**	**0.50**	0.25	**0.84**	-6.89
2q	1.31	1.47	4.21	**1.16**	1.11	0.24	1.58	**-2.93**
4q	2.76	2.73	9.25	**1.34**	2.30	0.23	3.10	**1.20**
8q	5.50	5.00	17.93	**1.87**	4.18	**0.23**	5.78	5.59
				BVAR(4)				
1q	**0.63**	0.67	1.90	1.11	0.52	**0.24**	0.85	**-6.93**
2q	**1.19**	1.29	3.90	1.24	**1.05**	**0.24**	**1.49**	-2.78
4q	**2.46**	2.41	8.35	1.49	**2.11**	**0.23**	**2.84**	1.49
8q	**4.93**	4.62	16.60	2.11	**3.87**	0.23	5.40	6.11
				DSGE				
1q	0.70	**0.62**	**1.77**	1.10	0.65	0.24	0.85	-6.82
2q	1.37	**1.21**	**3.64**	1.25	1.30	0.28	1.51	-2.33
4q	2.80	**2.18**	**7.72**	1.69	2.52	0.31	2.87	1.46
8q	5.27	**3.77**	**14.43**	2.86	4.53	0.37	**5.21**	**4.53**

Notes: The data starts in 1966:Q1 and the forecast period is 2005:Q1-2017:Q4 with all models being reestimated each quarter with an expanding data window. The overall measure is calculated as the log determinant of the uncentered forecast error covariance matrix as in Smets and Wouters (2007). Boldfaced numbers in red indicate best performance among the three models.

The result shows that our scenario analysis, utilizing future realized federal funds rates, generates more accurate predictions than unconditional forecast for

Table 2: Percentage gains(+) or losses(-) relative to unconditional forecast performance

Forecast horizon	GDP	CONS	INV	Wage	*l*Hours	*dl*P	Average
				VAR(1)			
1q	**-0.19**	**-3.61**	**-3.51**	**-0.72**	**-11.59**	1.78	**-2.96**
2q	0.57	**-1.23**	**-2.86**	**-2.16**	**-8.05**	0.47	**-2.51**
4q	5.09	2.80	2.34	**-0.61**	2.72	15.77	2.86
8q	6.83	5.19	5.53	**-0.74**	9.00	35.89	6.10
				BVAR(4)			
1q	1.71	1.35	**-1.87**	**-0.23**	**-6.07**	0.16	**-0.94**
2q	3.53	1.71	0.22	**-1.63**	**-2.79**	**-0.36**	0.28
4q	4.43	1.32	1.82	**-1.04**	3.56	13.34	2.29
8q	3.04	**-0.71**	1.82	**-1.78**	5.98	28.13	2.21
				DSGE			
1q	2.30	**-0.57**	1.34	0.13	4.87	1.26	1.45
2q	4.41	**-2.56**	5.54	0.44	8.05	**-1.89**	3.83
4q	5.40	**-8.75**	9.91	4.58	10.18	**-16.53**	6.30
8q	6.97	**-22.68**	16.98	13.50	9.50	**-43.08**	9.97

Notes: Boldfaced numbers in red indicate worse performance than unconditional forecast.

most variables in the three models, especially at longer horizons. This implies that all the models more or less reflect the true dynamics among most model variables, at least in qualitative sense. More specifically, when conditioning on the realized interest rate, the two VAR models provides substantially better forecast on inflation. However, their predictions for real wage seem to be worse than in the unconditional case. This suggests that these two models capture the comovement between interest rate and inflation quite well while missing out on the relationship between interest rate and wage. The DSGE model, in contrast, produces better forecast for wage under scenario analysis but its prediction for inflation is further away from the realized data when compared with the unconditional case. In addition, conditioning on interest rate also leads to deteriorated forecast for consumption, which suggests mis-specification in some aspect of the underlying dynamics. Finally, its forecast for GDP, investment and labor input all exhibit gains to various extents.

By comparing across the three models, we note that the improvements under scenario analysis are more significant for the DSGE model in forecasting GDP, investment, wage and hours worked. This suggests that the DSGE characterizes the comovements between interest rate and these four variables better than the other two models. Recall that in unconditional forecast, the BVAR model performs better than the DSGE model in predicting these four variables. This may be due to the DSGE's unsatisfactory forecast on the interest rate, as can be seen in Table 8.

When federal funds rate is controlled, performance of the DSGE model improves the most and becomes the best model in predicting investment over all horizons. Hence, good unconditional forecast performance does not necessarily imply a superior scenario analysis result. Comparison on the wage forecast provides further support on this argument. Despite the fact that the VAR(1) model provides the best unconditional forecast on wage and continues to be the best performing model on this variable under the scenario constraint (see Table 1), its prediction performance worsens when conditioning on the realized interest rate as reflected in Table 2. In contrast, the DSGE model gains the most on wage forecast under scenario analysis (see Table 2), but its prediction performance is still dominated by the other two models.

3.6 Subsample scenario analyses

To further understand the relative performance of the three models under different economic circumstances, we conduct the same back-testing exercise with four subsamples characterized by interest rate movement. Specifically, for each h-period ahead scenario analyses ($h = 1, 2, 4$ and 8) over 2005:Q1 to 2017:Q4, we calculate the magnitude of interest rate change at each starting time t as $|\Delta i_{t,h}| = |i_{t+h} - i_t|$, and partition the forecasting sequence into four groups depending on whether $|\Delta i_{t,h}|$ falls within 25, 50 or 75 percentile of the sequence $\{|\Delta i_{t,h}|, t =$ 2005:Q1, $\cdots$,2017:Q4-$h\}$. In other words, we are interested to know whether the models' performance is tied to the interest rate environment characterized by the magnitude of interest rate move over the forecast horizon. Tables 3-6 present the results for each subsample with Group 1 having the smallest interest rate change and Group 4 the largest. Boldfaced numbers in red indicate the best performer among the three models.

The results from Tables 3-6 suggest that during normal times when interest rates are relatively stable, the BVAR(4) is the best model in predicting GDP, investment and labor input conditional on realized interest rate, while the VAR(1) model provides the best forecast in wage and inflation and the DSGE model in consumption. However, as interest rates become more volatile, or in other words, there is large rate change over the forecast horizon, the DSGE model surpass the BVAR model for GDP and investment. For the ease of comparison, Figure 4 summarizes the results with the fill pattern in each cell indicating the best model (with the smallest RMSE) under scenario analysis with realized interest rates for different groups, forecast horizons, and variables of interest.

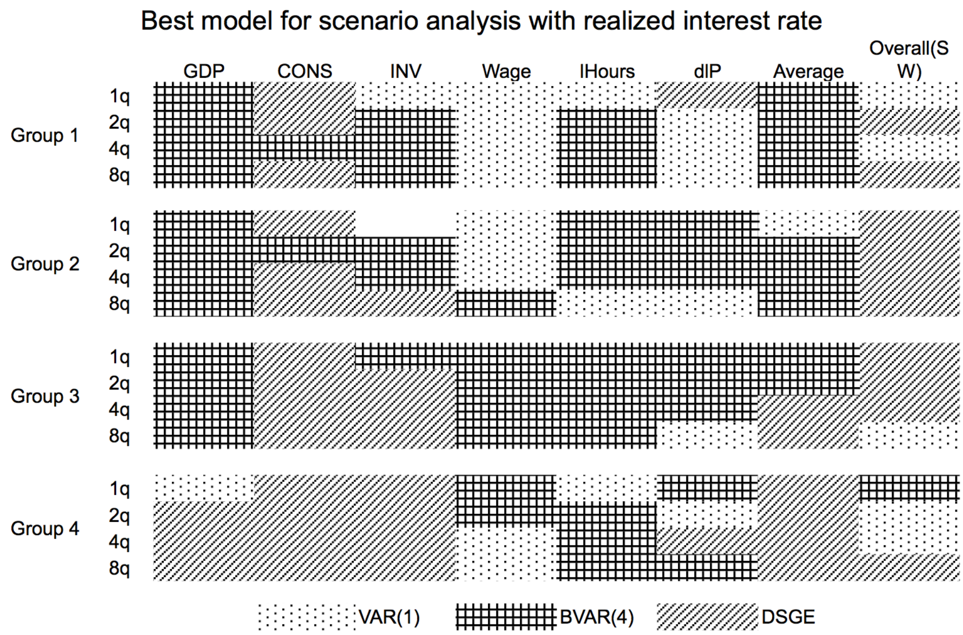


Figure 4: Back testing in scenario analysis for interest rate groups

Table 3: Back testing on scenario analysis – Group 1 (bottom quartile in interest rate movement)

Forecast horizon	GDP	CONS	INV	Wage	*l*Hours	*dl*P	Average	Overall (SW)*
				VAR(1)				
1q	0.49	0.50	**1.39**	**1.10**	**0.38**	0.33	0.70	-8.76
2q	0.92	0.85	2.03	**1.14**	0.43	**0.24**	0.93	-8.15
4q	1.92	1.95	5.50	**1.33**	1.69	**0.22**	2.10	**-4.96**
8q	5.05	4.58	7.10	**1.97**	2.75	**0.18**	3.61	-1.39
				BVAR(4)				
1q	**0.43**	0.37	1.40	1.21	0.43	0.33	**0.70**	**-8.77**
2q	**0.72**	0.62	**1.84**	1.18	**0.42**	0.27	**0.84**	-7.90
4q	**1.34**	**1.25**	**3.86**	1.73	**1.29**	0.24	**1.62**	-4.28
8q	**4.09**	3.76	**4.90**	2.75	**2.33**	0.21	**3.01**	-0.52
				DSGE				
1q	0.51	**0.37**	1.48	1.12	0.58	**0.29**	0.72	-8.10
2q	1.10	**0.56**	1.94	1.19	0.55	0.27	0.93	**-9.19**
4q	2.23	1.57	4.21	1.46	2.11	0.35	1.99	-4.04
8q	5.25	**3.29**	6.99	2.80	3.81	0.41	3.76	**-2.83**

Table 4: Back testing on scenario analysis – Group 2 (2nd quartile in interest rate movement)

Forecast horizon	GDP	CONS	INV	Wage	*l*Hours	*dl*P	Average	Overall (SW)*
				VAR(1)				
1q	0.45	**0.30**	0.82	**1.16**	0.43	0.24	**0.57**	-10.65
2q	0.89	1.01	2.51	**1.32**	1.12	0.21	1.18	-6.29
4q	2.45	2.16	3.41	**1.50**	1.10	0.16	1.80	-4.38
8q	4.23	3.31	7.85	1.64	2.30	**0.27**	3.27	-2.40
				BVAR(4)				
1q	**0.42**	0.37	0.84	1.26	**0.39**	**0.22**	0.58	-10.22
2q	**0.72**	**0.81**	**1.91**	1.58	**0.98**	**0.21**	**1.04**	-6.61
4q	**2.06**	1.75	**3.16**	1.64	**0.98**	**0.16**	**1.62**	-3.88
8q	**3.45**	2.37	6.57	**1.64**	**1.82**	0.28	**2.69**	-1.65
				DSGE				
1q	0.65	0.35	**0.81**	1.27	0.56	0.23	0.64	**-11.15**
2q	1.03	0.89	2.13	1.36	1.52	0.29	1.20	**-7.46**
4q	2.70	**1.72**	4.23	1.90	1.83	0.36	2.12	**-5.42**
8q	3.97	**2.16**	**5.61**	2.94	2.92	0.41	3.00	**-3.90**

Table 5: Back testing on scenario analysis – Group 3 (3rd quartile in interest rate movement)

Forecast horizon	GDP	CONS	INV	Wage	*l*Hours	*dl*P	Average	Overall (SW)*
				VAR(1)				
1q	0.67	0.69	1.67	**0.95**	0.53	0.21	0.79	-9.04
2q	1.24	1.30	4.56	**1.12**	1.23	0.34	1.63	-3.25
4q	1.83	1.75	8.30	1.14	2.13	0.21	2.56	-5.66
8q	4.49	3.94	17.93	1.81	3.57	**0.23**	5.33	**-3.42**
				BVAR(4)				
1q	**0.60**	0.59	**1.51**	0.97	**0.44**	**0.20**	**0.72**	-9.43
2q	**0.98**	0.85	4.22	1.13	**1.19**	**0.31**	**1.45**	-4.32
4q	**1.38**	1.07	6.95	**1.10**	**1.90**	**0.18**	2.10	-5.17
8q	**3.48**	3.08	15.53	**1.66**	**3.15**	0.24	4.52	-1.62
				DSGE				
1q	0.71	**0.57**	1.51	0.99	0.65	0.27	0.79	**-10.05**
2q	1.27	**0.82**	**3.75**	1.32	1.34	0.36	1.48	**-4.92**
4q	1.81	**0.91**	**5.70**	1.84	2.04	0.23	**2.09**	**-6.51**
8q	4.03	**2.50**	**12.38**	3.32	3.49	0.29	**4.33**	-3.25

Table 6: Back testing on scenario analysis – Group 4 (top quartile in interest rate movement)

Forecast horizon	GDP	CONS	INV	Wage	lHours	dlP	Average	Overall (SW)*
				VAR(1)				
1q	**0.83**	1.23	2.79	1.05	**0.62**	0.18	1.12	-10.28
2q	1.91	2.26	6.23	1.04	1.38	**0.16**	2.16	**-8.06**
4q	4.18	4.26	15.21	**1.35**	3.54	0.32	4.81	**-3.81**
8q	7.57	7.24	29.21	**2.03**	6.65	0.24	8.82	-2.81
				BVAR(4)				
1q	0.91	1.08	3.05	**0.99**	0.75	**0.17**	1.16	**-11.31**
2q	1.90	2.17	5.95	**0.98**	**1.34**	0.17	2.08	-7.89
4q	4.03	4.19	14.35	1.39	**3.41**	0.31	4.61	-2.71
8q	7.51	7.51	28.18	2.20	**6.43**	**0.20**	8.67	-2.50
				DSGE				
1q	0.86	**0.97**	**2.67**	1.03	0.77	0.17	**1.08**	-11.20
2q	**1.90**	**1.99**	**5.47**	1.12	1.49	0.21	**2.03**	-6.84
4q	**3.98**	**3.56**	**13.04**	1.53	3.67	**0.29**	**4.35**	-3.48
8q	**7.17**	**5.94**	**24.49**	2.30	6.85	0.36	**7.85**	**-3.35**

4 Conclusion

In this paper, we demonstrate that dynamic scenario analysis can provide more intuitive and informative answer to a large number of practical questions that are naturally appealing to policymakers, comparing to conventional forecasting exercise and structural (shock) analysis. Since it directly addresses situations that policy advisors recognize, the results can be easily interpreted to provide policy basis. In real life, policy authorities like central banks have 'beyond-model' information such as future macro objectives. Scenario analysis allows them to generate simulations conditional on these objectives or hypothetical stressed conditions to provide impact evaluation or early warnings. Furthermore, the simulated paths can be taken as a benchmark for policy makers to monitor whether they are on track to the target.

Technically, scenario analysis requires one to simulate the dynamic model forward while respecting restrictions imposed on some endogenous observables at a future time. Such bridge distribution is analytically solvable only for a few simple univariate processes such as AR(1). In macro applications, however, it is mostly intractable due to the high dimensionality and complexity of a model. All existing methods suffer from some kinds of inflexibility. Some are specially designed for VAR models, some can only deal with hard conditions, and others fail to consider parameter uncertainty. Our method provides an unified solution. It can flexibly deal with both hard and soft scenario settings while accounting for parameter uncertainty. More importantly, it can be extended to nonlinear and non-Gaussian models fairly straightforwardly.

The flexibility of our bridge-sampling method enables a scenario-based performance study of different models, which we choose to be the VAR(1), BVAR(4) and DSGE models developed in Smets and Wouters (2007). It is shown that these three models are comparable in terms of data likelihood and forecasting performance. But interestingly, they differ considerably under a scenario analysis using a well-known inflation targeting policy. The performance comparison between scenario analysis and the conventional forecast suggests that the two complementary dimensions are informative in distinguishing models. The DSGE model, which shows a relatively weak performance in conventional forecast over 2005 - 2017, has the greatest gains in performance for most variables when conditioning on the realized interest rate. Therefore, scenario analysis can provide insights into how a model performs in terms of characterizing the comovements between the conditioning and other variables. It also provides a new angle for model assessment in addition to historical data fit and unconditional forecasting performance.

A Density-Tempered SMC Bridge Sampler for Linear State-Space Model

We consider a generic linear state-space model with m-dimensional VAR(1) process driving the state variables $\boldsymbol{s}_t$ and n-dimensional observed variables $\boldsymbol{y}_t$ where $m \geq n$:

$$\begin{aligned} \boldsymbol{s}_t =& \boldsymbol{C} + \boldsymbol{B}\boldsymbol{s}_{t-1} + \boldsymbol{\Omega}\boldsymbol{\epsilon}_t^s \quad (7)\\ \boldsymbol{y}_t =& \boldsymbol{\mu} + \boldsymbol{M}\boldsymbol{s}_t + \boldsymbol{\epsilon}_t^y \quad (8)\\ \boldsymbol{\epsilon}_t^s \sim& D_s(\boldsymbol{0}, \boldsymbol{\Sigma}^s)\\ \boldsymbol{\epsilon}_t^y \sim& D_y(\boldsymbol{0}, \boldsymbol{\Sigma}^y) \end{aligned}$$

where $\boldsymbol{C}$ and $\boldsymbol{\mu}$ are vectors of dimension $m \times 1$ and $n \times 1$, respectively. $\boldsymbol{B}$ denotes the autoregressive matrix and $\boldsymbol{M}$ the observation selection matrix. The m-dimensional state $\boldsymbol{s}_t$ could be driven by a smaller number of shocks, as in most macro DSGE models including Smets and Wouters (2007). Suppose there are l shocks, then the matrix $\boldsymbol{\Omega}$ is of dimension $m \times l$ and its entries are some transformation of known model parameters. The l shocks are collected in $\boldsymbol{\epsilon}_t^s$, and they are *i.i.d.* over time, follow some distribution D_s with mean $\boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma}^s$, but need not be Gaussian random variables. Likewise is $\boldsymbol{\epsilon}_t^y$, which follows distribution D_y and has a covariance matrix denoted by $\boldsymbol{\Sigma}^y$. Furthermore, $\boldsymbol{\epsilon}_t^s$ and $\boldsymbol{\epsilon}_t^y$ are independent of each other, and their distributions may contain parameters beyond means and covariance matrix if non-Gaussian. Here we only consider the first lag term because any VAR(p) process can be straightforwardly converted into a higher-dimensional VAR(1) model.

Suppose at time T, we are interested in conducting a scenario analysis where the scenario imposes restrictions on the values of the first k observables τ period ahead, denoted by $\boldsymbol{y}_{T+\tau}^{(1:k)}$. The scenario condition can be written as $\boldsymbol{y}_{T+\tau}^{(1:k)} \in A \subseteq \boldsymbol{R}^k$, where $A = \times_{i=1}^k A_i$ and A_i corresponds to the constraint set for $\boldsymbol{y}_{T+\tau}^{(i)}$. In other words, given the observations up to period T, i.e., $\boldsymbol{y}_{1:T}$, we aim to generate bridge paths, $\boldsymbol{Y} = (\boldsymbol{y}_{T+1}, \boldsymbol{y}_{T+2}, \cdots, \boldsymbol{y}_{T+\tau})$, representing the conditional distribution $p(\boldsymbol{Y}|\boldsymbol{y}_{1:T}, \boldsymbol{y}_{T+\tau}^{(1:k)} \in A)$. Our method adopts the bridge-sampling technique proposed in Duan and Zhang (2016), adds parameter uncertainty and adapts it to the state-space setting.

By assuming that all parameters are subject to sampling errors, we need to consider an augmented space $(\boldsymbol{Y}, \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector containing all model parameters. To adapt to the state-space setting, we further extend the augmented space to include $\boldsymbol{s}_{T:(T+\tau)}$ because at each t, we need both observables $\boldsymbol{y}_t$ and unobservables $\boldsymbol{s}_t$ to provide sufficient information to roll the model forward. However, after getting $\boldsymbol{s}_T$, by equation (7), $\boldsymbol{s}_t$ and $\epsilon_{s,t}$ carry the same information for t running from $T+1$ to $T+\tau$. Hence one can express the augmented system in either $\boldsymbol{s}_{T:(T+\tau)}$ or $(\boldsymbol{E}_s, \boldsymbol{s}_T)$, where $\boldsymbol{E}_s = \boldsymbol{\epsilon}^s_{(T+1):(T+\tau)}$, but it turns out that the latter is more convenient to work with in dealing with degeneracy due to $l < m$ in models such as Smets and Wouters (2007) to be elaborated later in subsection A.5. So, the complete augmented system is considered to be$(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})$. Mathematically, we define the logarithm of the target bridge density as follows:

$$\begin{aligned}
&\ln \mathcal{L}_{TBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \\
=&\ln p(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta} | \boldsymbol{y}_{1:T}, \boldsymbol{y}_{T+\tau}^{(1:k)} \in A) \\
=&\ln p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T}) + \ln f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T}) + \ln f_D(\boldsymbol{Y}, \boldsymbol{E}_s | \boldsymbol{s}_T, \boldsymbol{\theta}, \boldsymbol{y}_{1:T}) \\
&- \ln f_D(\boldsymbol{y}_{T+\tau}^{(1:k)} \in A | \boldsymbol{y}_{1:T}) \\
=&\ln p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T}) + \ln f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T}) + \sum_{t=T+1}^{T+\tau} \ln f_D(\boldsymbol{y}_t, \boldsymbol{\epsilon}_t^s | \boldsymbol{s}_{t-1}, \boldsymbol{\theta}) \\
&- \ln f_D(\boldsymbol{y}_{T+\tau}^{(1:k)} \in A | \boldsymbol{y}_{1:T})
\end{aligned} \tag{9}$$

where $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ is the parameter density given observations up to time T, the filtering density for $\boldsymbol{s}_T$ is $\ln f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T})$, and $\ln f_D(\boldsymbol{y}_{T+\tau}^{(1:k)} \in A|\boldsymbol{y}_{1:T})$ is the probability of the scenario conditional on $\boldsymbol{y}_{1:T}$. f_D indicates that the density is evaluated with respect to the target state-space model given by equations (7)-(8). Note that if $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ is derived from a frequentist approach, it is typically an asymptotic distribution of a known form. Under the Bayesian analysis using MCMC or SMC, $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ is represented by a sample. This is actually an important factor to our algorithm design which has broader applications in mind. The method proposed here does not need to know its analytical form of $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$.

A.1 Initialization

The target bridge density function in most cases lacks analytical formula and hence cannot be sampled directly. Following Duan and Zhang (2016), we apply the idea of sequential importance sampling, and start with a sample from some easy-to-draw bridge distribution while knowing it is incorrect. Below we design an efficient generator for the initialization.

Under the augmented space, we need to generate initialization for $\boldsymbol{Y}$, $\boldsymbol{E}_s$, $\boldsymbol{s}_T$ and $\boldsymbol{\theta}$. Sampling $\boldsymbol{\theta}$ and $\boldsymbol{s}_T$ is straightforward. Either $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ is an asymptotic distribution of a known form which we can sample accordingly or we will have a sample directly from the Bayesian estimation step. If we are not confident about the quality of the sample from the Bayesian estimation step in representing $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$, or the scenario imposed is so extreme that we feel the sample does not provide an adequate empirical support to cover the parameter distribution after factoring in the scenario condition, we can choose to re-initialize $\boldsymbol{\theta}$ by a Gaussian density approximating $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$, with an enlarged variance if necessary. We denote this Gaussian density by $I(\theta)$, and it is used as the generator for $\boldsymbol{\theta}$ in our subsequent illustration unless specified otherwise. When latency is involved, $\boldsymbol{s}_T$ can be sampled according to the filtering distribution $f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T})$ which is also available from the Bayesian or frequentist estimation step.

For initialization of $\boldsymbol{Y}$, we proceed in two steps. We first partition the observation vector into two groups: $\boldsymbol{y}_t = (\boldsymbol{y}_t^{(1:k)}, \boldsymbol{y}_t^{(k+1:n)})$ where the conditioning variables are gathered in the first k elements. We further denote the bridge paths for conditioning variables by $\boldsymbol{Y}_{1:k} = \boldsymbol{y}^{(1:k)}_{T+1:T+\tau}$ and those of companion observations by $\boldsymbol{Y}_{k+1:n} = \boldsymbol{y}^{(k+1:n)}_{T+1:T+\tau}$. In the first step, we sample bridge paths for conditioning variables $\boldsymbol{Y}_{1:k}$, one variable at a time based on the estimated univariate AR(1) process. In the second step, we generate bridge paths for the other companion variables conditional on $\boldsymbol{Y}_{1:k}$ via a simple decomposition. To be more specific, we assume each of the conditioning variables $y_{i,t}$, where $i = 1 : k$, follows a Gaussian AR(1) process, knowing that they may not be true but can be compensated with an importance weight in sampling later. That is, we estimate the following assumed AR(1) process using the observed data.

$$\begin{aligned} y_{i,t} =& \mu_i + \rho_i y_{i,t-1} + \epsilon_{it} \\ \epsilon_{it}|\mathcal{F}_{t-1} \sim& N(0, \sigma_i^2). \end{aligned} \tag{10}$$

where μ_i is a constant scalar and ρ_i is the autoregressive coefficient. Innovation ϵ_{it} conditional on information set $\mathcal{F}_{t-1}$ is assumed Gaussian with mean 0 and variance σ_i^2.

Before sampling the bridge paths for $y_{i,t}$, one needs to fix the value of its endpoint $y_{i,(T+\tau)}$. The sampling of the endpoint will depend on the type of condition imposed on $y_{i,(T+\tau)}$, i.e., the structure of A_i. If $y_{i,(T+\tau)}$ is constrained by a hard condition, i.e. A_i is a singleton, $y_{i,(T+\tau)}$ will simply equal the only permissible endpoint. On the other hand, if policy makers want to allow for some uncertainty on the conditioning information so that the scenario requires the endpoint to fall within some range, we will first draw $y_{i,(T+\tau)}$ with some distribution $g_i(y_{i,(T+\tau)}|y_{i,T}, A_i)$ based on the estimated AR(1) process, which we will discuss in a later subsection.

Once $y_{i,(T+\tau)}$ is drawn, we can generate bridge paths according to the explicit representation of the Gaussian AR(1) bridge conditional on a fixed endpoint:

$$\begin{aligned} y_{i,t} =& \alpha_{it}(y_{i,t-1}) + \beta_{it} y_{i,(T+\tau)} + u_{it} \\ \alpha_{it}(y_{i,t-1}) =& \mu_i + \rho_i y_{i,t-1} - \beta_{it}\left(\frac{1-\rho_i^{(T+\tau-t+1)}}{1-\rho_i}\mu_i + \rho_i^{(T+\tau-t+1)} y_{i,t-1}\right) \\ \beta_{it} =& \rho_i^{(T+\tau-t)} \frac{1-\rho_i^2}{1-\rho_i^{2(T+\tau-t+1)}} \\ \zeta_{it}^2 =& \frac{1-\rho_i^2 - \beta_{it}^2(1-\rho_i^{2(T+\tau-t+1)})}{1-\rho_i^2}\sigma_i^2 \\ u_{it} \sim& N(0, \zeta_{it}^2) \quad i.i.d. \end{aligned} \tag{11}$$

where μ_i, ρ_i, and σ_i are estimated using equation (10); $\alpha_{it}(y_{i,t-1})$ and β_{it} come from the standard Gaussian bridge results.

With the bridge for conditioning variables in place, we can then generate other variables in the state-space model by utilizing the following equation deduced from equations (7)-(8) and assuming that $\boldsymbol{\epsilon}_t^s$ and $\boldsymbol{\epsilon}_t^y$ are Gaussian for the generator.

$$\begin{aligned} \boldsymbol{y}_t =& \boldsymbol{\mu}^* + \boldsymbol{M}^* \boldsymbol{s}_{t-1} + \boldsymbol{\epsilon}_t^{y*} \\ \boldsymbol{\epsilon}_t^{y*} \sim& N(\boldsymbol{0}, \boldsymbol{\Sigma}^{y*}) \end{aligned} \tag{12}$$

where $\boldsymbol{\mu}^* = \boldsymbol{\mu} + \boldsymbol{M}\boldsymbol{C}$, $\boldsymbol{M}^* = \boldsymbol{M}\boldsymbol{B}$, $\boldsymbol{\epsilon}_t^{y*} = \boldsymbol{M}\boldsymbol{\Omega}\boldsymbol{\epsilon}_t^s + \boldsymbol{\epsilon}_t^y$, and $\boldsymbol{\Sigma}^{y*} = \boldsymbol{M}\boldsymbol{\Omega}\boldsymbol{\Sigma}^s\boldsymbol{\Omega}'\boldsymbol{M}' + \boldsymbol{\Sigma}^y$. Express with sub-blocks: $\boldsymbol{\mu}^* = \begin{bmatrix} \boldsymbol{\mu}_k^* \\ \boldsymbol{\mu}_{k-}^* \end{bmatrix}$, $\boldsymbol{M}^* = \begin{bmatrix} \boldsymbol{M}_k^* \\ \boldsymbol{M}_{k-}^* \end{bmatrix}$, and $\boldsymbol{\Sigma}^{y*} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{y*} & \boldsymbol{\Sigma}_{12}^{y*} \\ \boldsymbol{\Sigma}_{21}^{y*} & \boldsymbol{\Sigma}_{22}^{y*} \end{bmatrix}$. Then one can easily derive

$$E(\boldsymbol{y}_t^{(k+1:n)}|\boldsymbol{y}_t^{(1:k)}, \boldsymbol{s}_{t-1}, \boldsymbol{\theta}) = \boldsymbol{\mu}_{k-}^* + \boldsymbol{M}_{k-}^*\boldsymbol{s}_{t-1} + \boldsymbol{\Sigma}_{21}^{y*}\left(\boldsymbol{\Sigma}_{11}^{y*}\right)^{-1}\left(\boldsymbol{y}_t^{(1:k)} - \boldsymbol{\mu}_k^* - \boldsymbol{M}_k^*\boldsymbol{s}_{t-1}\right) \tag{13}$$

$$Var(\boldsymbol{y}_t^{(k+1:n)}|\boldsymbol{y}_t^{(1:k)}, \boldsymbol{s}_{t-1}, \boldsymbol{\theta}) = \boldsymbol{\Sigma}_{22}^{y*} - \boldsymbol{\Sigma}_{21}^{y*}\left(\boldsymbol{\Sigma}_{11}^{y*}\right)^{-1}\boldsymbol{\Sigma}_{12}^{y*}. \tag{14}$$

and use these two sufficient statistics to generate paths for $\boldsymbol{y}_t^{(k+1:n)}$, the other observable variables.

Of course, we also need to generate the state variables, $\boldsymbol{s}_t$, conditional on $\boldsymbol{y}_t$, because they form part of the information set for advancing the system to the next time point. For this, we first generate $\boldsymbol{\epsilon}_t^s$ according to the following distribution:

$$\begin{aligned}
E(\boldsymbol{\epsilon}_t^s|\boldsymbol{y}_t, \boldsymbol{s}_{t-1}, \boldsymbol{\theta}) &= E(\boldsymbol{\epsilon}_t^s|\boldsymbol{\epsilon}_t^{y*}, \boldsymbol{\theta}) \\
&= \boldsymbol{\Sigma}^s\boldsymbol{\Omega}'\boldsymbol{M}'\left(\boldsymbol{\Sigma}^{y*}\right)^{-1}\boldsymbol{\epsilon}_t^{y*} \\
&= \boldsymbol{\Sigma}^s\boldsymbol{\Omega}'\boldsymbol{M}'\left(\boldsymbol{\Sigma}^{y*}\right)^{-1}\left(\boldsymbol{y}_t - \boldsymbol{\mu}^* - \boldsymbol{M}^*\boldsymbol{s}_{t-1}\right) \qquad (15)\\
Var(\boldsymbol{\epsilon}_t^s|\boldsymbol{y}_t, \boldsymbol{s}_{t-1}, \boldsymbol{\theta}) &= Var(\boldsymbol{\epsilon}_t^s|\boldsymbol{\epsilon}_t^{y*}, \boldsymbol{\theta}) \\
&= \boldsymbol{\Sigma}^s - \boldsymbol{\Sigma}^s\boldsymbol{\Omega}'\boldsymbol{M}'\left(\boldsymbol{\Sigma}^{y*}\right)^{-1}\boldsymbol{M}\boldsymbol{\Omega}\left(\boldsymbol{\Sigma}^s\right). \qquad (16)
\end{aligned}$$

This gives our initialization for $\boldsymbol{\epsilon}_t^s$ and also completes our initialization for the entire augmented system. After drawing $\boldsymbol{\epsilon}_t^s$, $\boldsymbol{s}_t$ can be updated accordingly using the transition equation (7).

If the target model is VAR(1) instead of the state-space model, the method described above can be greatly simplified, because latency is absent from the model. Specifically, we no longer need the filtering distribution to sample the initial latent variables. The conditioning variable $\boldsymbol{s}_{t-1}$ appearing in many places should also be replaced by $\boldsymbol{y}_{t-1}$, and equation (12) is used instead. Moreover, equations (15)-(16) used for sampling $\boldsymbol{\epsilon}_t^s$ are no longer required.

The overall density of this initialization bridge sampler is the product of (1) the initialization density for the model parameter, i.e., $I(\boldsymbol{\theta})$, which is by choice a Gaussian density approximating $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$, (2) the filtering distribution for $\boldsymbol{s}_T$ based on the state-space model in equations (7)-(8), i.e., $f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T})$, (3) the density of the AR(1) bridge for each of the k conditioning variables, which is Gaussian by design, (4) the conditional density of the bridge paths for the remaining observables which is again Gaussian by design, and finally (5) the density of $\boldsymbol{\epsilon}_t^s$ conditional on $\boldsymbol{y}_t$.

Since each conditioning variable $y_{i,t}$, where $i = 1 : k$, is assumed to follow the Gaussian bridge process defined in equation (11), its density only depends on the estimated coefficients for their respective AR(1) processes and $y_{i,T}$, which can be written as:

$$
\begin{aligned}
&\ln \mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T)\\
=&\sum_{i=1}^{k}\left[\sum_{t=T+1}^{T+\tau-1}\ln f(y_{i,t}|y_{i,t-1},y_{i,T+\tau})+\ln g_i(y_{i,T+\tau}|y_{i,T},A_i)\right]
\end{aligned}
\tag{17}
$$

where

$$
\ln f(y_{i,t}|y_{i,t-1},y_{i,T+\tau}) = -\frac{1}{2}\ln(2\pi\zeta_{it}^2) - \frac{(y_{i,t}-\alpha_{it}(y_{i,t-1})-\beta_{it}y_{i,T+\tau})^2}{2\zeta_{it}^2}. \tag{18}
$$

Next, the generator for $\boldsymbol{Y}_{k+1:n}$ and $\boldsymbol{E}_s$ makes use of a decomposed form of equation (12), which depends on the initial state $\boldsymbol{s}_T$, the model parameter $\boldsymbol{\theta}$ and the bridge path for conditioning variables $\boldsymbol{Y}_{1:k}$. Hence, the logarithm of their joint density is:

$$
\begin{aligned}
&\ln \mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n},\boldsymbol{E}_s|\boldsymbol{Y}_{1:k},\boldsymbol{s}_T,\boldsymbol{\theta})\\
=&\sum_{t=T+1}^{T+\tau}\left[\ln f_G(\boldsymbol{y}_t^{(k+1:n)}|\boldsymbol{y}_t^{(1:k)},\boldsymbol{s}_{t-1},\boldsymbol{\theta})+\ln f_G(\boldsymbol{\epsilon}_t^s|\boldsymbol{y}_t,\boldsymbol{s}_{t-1},\boldsymbol{\theta})\right]
\end{aligned}
\tag{19}
$$

where $f_G(\boldsymbol{y}_t^{(k+1:n)}|\boldsymbol{y}_t^{(1:k)},\boldsymbol{s}_{t-1},\boldsymbol{\theta})$ is a Gaussian distribution with mean and variance given in equations (13) and (14). Similarly, $f_G(\boldsymbol{\epsilon}_t^s|\boldsymbol{y}_t,\boldsymbol{s}_{t-1},\boldsymbol{\theta})$ is Gaussian with mean and variance defined in equations (15) and (16). Notice that in the degenerate case such as the DSGE model of Smets and Wouters (2007), this density will become a Dirac delta function, but does not present a numerical difficulty to our bridge sampling method, which will become clear later.

In summary, the log-density of our initialization sampler can be written as:

$$
\begin{aligned}
&\ln \mathcal{L}_{IBridge}(\boldsymbol{Y},\boldsymbol{E}_s,\boldsymbol{s}_T,\boldsymbol{\theta})\\
=&\ln I(\boldsymbol{\theta})+\ln f_D(\boldsymbol{s}_T|\boldsymbol{\theta},\boldsymbol{y}_{1:T})+\ln \mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T)\\
&+\ln \mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n},\boldsymbol{E}_s|\boldsymbol{Y}_{1:k},\boldsymbol{s}_T,\boldsymbol{\theta}).
\end{aligned}
\tag{20}
$$

A.2 Importance weight and density tempering

Recall that our target process is a state-space model specified in equations (7)-(8), and the target bridge density is defined in equation (9). Following the theory of importance sampling, the correct importance weight assigned to the sample $(\boldsymbol{Y},\boldsymbol{E}_s,\boldsymbol{s}_T,\boldsymbol{\theta})$ equals:

$$
\begin{aligned}
&\frac{\mathcal{L}_{TBridge}(\boldsymbol{Y},\boldsymbol{E}_s,\boldsymbol{s}_T,\boldsymbol{\theta})}{\mathcal{L}_{IBridge}(\boldsymbol{Y},\boldsymbol{E}_s,\boldsymbol{s}_T,\boldsymbol{\theta})}\\
\propto&\frac{p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})\prod_{t=T+1}^{T+\tau}f_D(\boldsymbol{y}_t,\boldsymbol{\epsilon}_t^s|\boldsymbol{s}_{t-1},\boldsymbol{\theta})}{I(\boldsymbol{\theta})\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T)\mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n},\boldsymbol{E}_s|\boldsymbol{Y}_{1:k},\boldsymbol{s}_T,\boldsymbol{\theta})}.
\end{aligned}
\tag{21}
$$

The proportional sign holds because the $f_D(\boldsymbol{y}_{T+\tau}^{(1:k)}\in A|\boldsymbol{y}_{1:T})$ term in our target density is the probability of landing in the target set A averaged over all values of the latent variables using the filtering distribution at time T and over all parameter values using $p(\boldsymbol{\theta})$. It remains a constant for all samples in the augmented space and hence irrelevant. Furthermore, the filtering density $f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T})$ drops out of the importance weight formula completely through cancellation, therefore we do not need to know its analytical form.

Since the model from which we generate our initialization sample differs from the target state-space model, the importance weights across sample points are expected to be quite uneven. If we directly resample according to these weights, only a few or a small fraction of the original sample points will be retained, resulting in severe empirical support shrinkage and sample impoverishment. Hence, we must temper the importance weights sequentially to close the gap between the initialization generator and the target model. Following Duan and Zhang (2016), we smooth the transition by artificially constructing a series of intermediate target distributions:

$$\begin{aligned} & f_{\delta_i}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \\ \propto & \mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \times \left(\frac{\mathcal{L}_{TBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}{\mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})} \right)^{\delta_i} \\ & 0 = \delta_0 < \delta_1 < \delta_2 < \cdots = 1 \end{aligned} \tag{22}$$

By doing this, we break the transition task into a number of smaller movements. Each time when we proceed from δ_i to δ_{i+1}, we are one step closer to our target bridge distribution. In this sub-movement, $f_{\delta_i}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})$ can be viewed as an initialization bridge distribution, and $f_{\delta_{i+1}}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})$ our target distribution. Moving to the new target hence can be implemented by reweighting the paths with the following incremental importance weights:

$$\begin{aligned} W(\delta_i, \delta_{i+1}, \boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) &= \frac{f_{\delta_{i+1}}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}{f_{\delta_i}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})} \\ \propto & \frac{\mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \left(\frac{\mathcal{L}_{TBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}{\mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})} \right)^{\delta_{i+1}}}{\mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \left(\frac{\mathcal{L}_{TBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}{\mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})} \right)^{\delta_i}} \\ = & \left(\frac{\mathcal{L}_{TBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}{\mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})} \right)^{\delta_{i+1} - \delta_i} \\ \propto & \left(\frac{p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T}) \prod_{t=T+1}^{T+\tau} f_D(\boldsymbol{y}_t, \boldsymbol{\epsilon}_t^s | \boldsymbol{s}_{t-1}, \boldsymbol{\theta})}{I(\boldsymbol{\theta}) \mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T) \mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n}, \boldsymbol{E}_s | \boldsymbol{Y}_{1:k}, \boldsymbol{s}_T, \boldsymbol{\theta})} \right)^{\delta_{i+1} - \delta_i} \end{aligned} \tag{23}$$

and resampling accordingly.

The degree of sample impoverishment is directly linked to the unevenness of importance weights, hence following the convention in the SMC literature, we use the effective sample size (ESS) to monitor the diversity of paths. Denote our intended sample size by N, ESS as a percentage of N is calculated as:

$$ESS/N = \frac{1}{N \sum_{p=1}^{N_d} w_p^2} \tag{24}$$

where p is the index for particles, N_d is the number of distinct points and w_p are the importance weights converted into probabilities. If there are multiple duplicates of a particle in the sample, w_p is the sum of probabilities of all duplicates. To serve the purpose of maintaining good sample diversity, the series $\{\delta_1, \delta_2, \ldots\}$ is adaptively chosen such that the ESS stays above a certain level. Unless otherwise stated, we set the threshold at 50% of the total sample size. To be more specific, with a sample representing $f_{\delta_i}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})$, we calculate the incremental importance weights W and ESS/N according to equations (23) and (24) for various values of $\delta_{i+1} \in (\delta_i, 1]$. We pick the maximum δ_{i+1} that satisfies $ESS/N \geq 0.5$, and that determines our next target distribution. By resampling and reassigning equal weights to the simulated paths, we advance bridge paths to $f_{\delta_{i+1}}$, and can proceed to choose the next target in the sequence.

A.3 Support boosting

It has been elaborated in Duan and Zhang (2016) that periodically boosting the empirical support is essential for the bridge-sampling scheme to succeed, and the Metropolis-Hastings (MH) move can help accomplish this task. The MH move starts with a newly proposed particle $(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)$ in the augmented space, which is used to replace the original point $(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})$ at some acceptance rate determined by the quality of the proposal.

One natural proposal for the MH move is again the initialization sampler. Here we make minor modifications to make it adaptive and more efficient. Recall that our initialization for model parameters is $\boldsymbol{I}(\boldsymbol{\theta})$ which is a Gaussian density approximating $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$. In the MH move, we consider a mixed proposal for $\boldsymbol{\theta}^*$ denoted by $h(\boldsymbol{\theta}^*|\boldsymbol{\theta})$, which is a 50-50 mixture of independent and symmetric random walk proposals, and both are Gaussian. The independent proposal is intended to approximate the available SMC sample of $\boldsymbol{\theta}$ at the stage, i.e., using the sample's mean vector and covariance matrix, whereas the symmetric random walk proposal uses a scaled-down covariance matrix (20% of the sample standard deviations) to provide some local variation. $\boldsymbol{s}_T^*$ is again sampled from the filtering distribution $f_D(\boldsymbol{s}_t^*|\boldsymbol{\theta}^*, \boldsymbol{y}_{1:T})$ based on the state-space model in equations (7)-(8), but at a different parameter value $\boldsymbol{\theta}^*$.

We use the same generating model given in equations (10) - (16) as the proposal for $\boldsymbol{Y}^*$ and $\boldsymbol{E}_s^*$ conditional on $\boldsymbol{s}_T^*$ and $\boldsymbol{\theta}^*$. The only difference from the initialization is that here we make use of a nice feature in Duan and Zhang (2016), which is to replace random segments of a bridge path instead of the entire path. For details on how to generate random starting and ending times with $(T+1) \leq t_s \leq t_e \leq (T+\tau-1)$, readers are referred to Duan and Zhang (2016). It is worth mentioning that there is a Bernoulli random variable Z with $p(Z=1) = 0.5$ which indicates whether the endpoints $\boldsymbol{y}_{T+\tau}^{(1:k)}$ is resampled. When $Z=1$, t_e is always set equal to $T+\tau-1$ and when $Z=0$, we set t_s equal to $T+1$ with 50% probability. With Z, t_s and t_e in place, we first generate the defiined segment of bridge paths for the k conditioning variables, again one at a time based on their respective estimated AR(1) processes in equation (10). Note that the new paths for conditioning variables $\boldsymbol{Y}_{1:k}^*$ are different from $\boldsymbol{Y}_{1:k}$ only over the segment defined by $[t_s, t_e]$ which is tied to Z. Hence, its log-density given current path $\boldsymbol{Y}$, while conditioning on $\boldsymbol{s}_T^*$, $\boldsymbol{\theta}^*$, Z, t_s and t_e is:

$$
\begin{aligned}
&\ln \mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{Y}, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*, Z, t_s, t_e) \\
=&\ln \mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{Y}, Z, t_s, t_e) \\
=&\begin{cases}
\sum_{i=1}^{k}\left[\sum_{t=t_s}^{t_e} \ln f(y_{i,t}^*|y_{i,t-1}^*, y_{i,t_e+1}, t_e)\right] & if\ Z=0 \\
\\
\sum_{i=1}^{k}\left[\sum_{t=t_s}^{T+\tau-1} \ln f(y_{i,t}^*|y_{i,t-1}^*, y_{i,T+\tau}^*) + \ln g_i(y_{i,T+\tau}^*|y_{i,t_s-1}, A_i)\right] & if\ Z=1
\end{cases}
\end{aligned}
$$

where $g_i(y_{i,T+\tau}^*|y_{i,t_s-1}, A_i)$ is the same endpoint sampler as in the initialization which will be described in the next subsection, and

$$
\ln f(y_{i,t}^*|y_{i,t-1}^*, y_{i,t_e+1}, t_e) = -\frac{1}{2}\ln\left(2\pi\zeta_{it}^2(t_e)\right) - \frac{\left(y_{i,t}^* - \alpha_{it}(y_{i,t-1}^*, t_e) - \beta_{it}(t_e)y_{i,t_e+1}\right)^2}{2\zeta_{it}^2(t_e)}
$$

where

$$
\begin{aligned}
\alpha_{it}(y_{i,t-1}^*, t_e) =&\mu_i + \rho_i y_{i,t-1}^* - \beta_{it}(t_e)\left(\frac{1-\rho_i^{(t_e-t+2)}}{1-\rho_i}\mu_i + \rho_i^{(t_e-t+2)}y_{i,t-1}^*\right) \\
\beta_{it}(t_e) =&\rho_i^{(t_e-t+1)}\frac{1-\rho_i^2}{1-\rho_i^{2(t_e-t+2)}} \\
\zeta_{it}^2(t_e) =&\frac{1-\rho_i^2-\beta_{it}(t_e)^2\left(1-\rho_i^{2(t_e-t+2)}\right)}{1-\rho_i^2}\sigma_i^2.
\end{aligned}
$$

Next, we use again equations (12) - (16) to sample $\boldsymbol{Y}_{k+1:n}^*$ and $\boldsymbol{E}_s^*$ conditional on $\boldsymbol{Y}_{1:k}^*$, $\boldsymbol{s}_T^*$ and $\boldsymbol{\theta}^*$. It is worth noting that $\boldsymbol{Y}_{k+1:n}^*$ and $\boldsymbol{E}_s^*$ will differ from $\boldsymbol{Y}_{k+1:n}$ and $\boldsymbol{E}_s$ at all time points from $T+1$ to $T+\tau$ even though $\boldsymbol{Y}_{1:k}^*$ and $\boldsymbol{Y}_{1:k}$ are identical before t_s and after t_e. This is because the distribution of $\boldsymbol{y}_t^{*(k+1:n)}$ depends on the previous state $\boldsymbol{s}_{t-1}^*$, and $\boldsymbol{s}_t^*$ starts to differ from $\boldsymbol{s}_t$ right from the beginning at time T. Therefore, the logarithm of the joint density of $\boldsymbol{Y}_{k+1:n}^*$ and $\boldsymbol{E}_s^*$ is:

$$
\begin{aligned}
&\ln \mathcal{L}_{MH}(\boldsymbol{Y}_{k+1:n}^*, \boldsymbol{E}_s^*|\boldsymbol{Y}_{1:k}^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*) \\
=&\sum_{t=T+1}^{T+\tau}\left[\ln f_G(\boldsymbol{y}_t^{*(k+1:n)}|\boldsymbol{y}_t^{*(1:k)}, \boldsymbol{s}_{t-1}^*, \boldsymbol{\theta}^*) + \ln f_G(\boldsymbol{\epsilon}_t^{s*}|\boldsymbol{y}_t^*, \boldsymbol{s}_{t-1}^*, \boldsymbol{\theta}^*)\right].
\end{aligned}
$$

Notice that this is actually the same as $\ln \mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n}^*, \boldsymbol{E}_s^*|\boldsymbol{Y}_{1:k}^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)$.

The overall density of the new particle $(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)$ under the MH proposal is then given by:

$$
\begin{aligned}
&\mathcal{L}_{MH}(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*|\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \\
=&h(\boldsymbol{\theta}^*|\boldsymbol{\theta})f_D(\boldsymbol{s}_T^*|\boldsymbol{\theta}^*, \boldsymbol{y}_{1:T})\mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{Y}, Z, t_s, t_e)\mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n}^*, \boldsymbol{E}_s^*|\boldsymbol{Y}_{1:k}^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)
\end{aligned}
$$

Suppose we are at the intermediate target f_{δ_i}, given a particle $(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})$, the replacement proposal $(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)$ should be accepted at the probability calculated as follows:

$$
\begin{aligned}
& a\{(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \rightarrow (\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)\} \\
= & \min\left(1, \frac{f_{\delta_i}(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)}{f_{\delta_i}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})} \frac{\mathcal{L}_{MH}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta} | \boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)}{\mathcal{L}_{MH}(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^* | \boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}\right)
\end{aligned} \tag{25}
$$

Note that $\mathcal{L}_{MH}(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^* | \boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})$ is similar to the initialization sampler, $\mathcal{L}_{IBridge}(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)$ in equation (20). The differences are (1) $h(\boldsymbol{\theta}^*|\boldsymbol{\theta})$ takes the place of $I(\boldsymbol{\theta}^*)$, and (2) $\mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{Y}, Z, t_s, t_e)$ takes the place of $\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{y}_{1:T})$. Thus, many terms simply drop out of the expression for the acceptance probability due to cancellation. That is,

$$
\begin{aligned}
& a\{(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \rightarrow (\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)\} \\
= & \min\left(1, \frac{\frac{I(\boldsymbol{\theta}^*)\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{y}_T)}{h(\boldsymbol{\theta}^*|\boldsymbol{\theta})\mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{Y}, Z, t_s, t_e)} \left(\frac{\mathcal{L}_{TBridge}(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)}{\mathcal{L}_{IBridge}(\boldsymbol{Y}^*, \boldsymbol{E}_s^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)}\right)^{\delta_i}}{\frac{I(\boldsymbol{\theta})\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T)}{h(\boldsymbol{\theta}|\boldsymbol{\theta}^*)\mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}|\boldsymbol{Y}^*, Z, t_s, t_e)} \left(\frac{\mathcal{L}_{TBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}{\mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta})}\right)^{\delta_i}}\right) \\
= & \min\left(1, \frac{\frac{I(\boldsymbol{\theta}^*)\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{y}_T)}{h(\boldsymbol{\theta}^*|\boldsymbol{\theta})\mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{Y}, Z, t_s, t_e)} \left(\frac{p(\boldsymbol{\theta}^*|\boldsymbol{y}_{1:T}) \prod_{t=T+1}^{T+\tau} f_D(\boldsymbol{y}_t^*, \boldsymbol{\epsilon}_{s,t}^* | \boldsymbol{s}_{t-1}^*, \boldsymbol{\theta}^*)}{I(\boldsymbol{\theta}^*)\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}^*|\boldsymbol{y}_T)\mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n}^*, \boldsymbol{E}_s^* | \boldsymbol{Y}_{1:k}^*, \boldsymbol{s}_T^*, \boldsymbol{\theta}^*)}\right)^{\delta_i}}{\frac{I(\boldsymbol{\theta})\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T)}{h(\boldsymbol{\theta}|\boldsymbol{\theta}^*)\mathcal{L}_{MH}(\boldsymbol{Y}_{1:k}|\boldsymbol{Y}^*, Z, t_s, t_e)} \left(\frac{p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T}) \prod_{t=T+1}^{T+\tau} f_D(\boldsymbol{y}_t, \boldsymbol{\epsilon}_t^s | \boldsymbol{s}_{t-1}, \boldsymbol{\theta})}{I(\boldsymbol{\theta})\mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T)\mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n}, \boldsymbol{E}_s | \boldsymbol{Y}_{1:k}, \boldsymbol{s}_T, \boldsymbol{\theta})}\right)^{\delta_i}}\right).
\end{aligned} \tag{26}
$$

The lack of an analytical form for $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ can be dealt with in two ways. First, choose both $h(\boldsymbol{\theta}^*|\boldsymbol{\theta})$ and $I(\boldsymbol{\theta}^*)$ to be the same as $p(\boldsymbol{\theta}^*|\boldsymbol{y}_{1:T})$ to completely cancel it out. Alternatively, numerically evaluate the posterior distribution by its definition:

$$
p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T}) \propto Prior(\boldsymbol{\theta})\mathcal{L}(\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_T|\boldsymbol{\theta}).
$$

Naturally, the first way is much simpler and entails lower computing costs, but the second approach may be necessary when the contemplated scenario is rather extreme. When $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ is represented by a sample, repeatedly generating $\boldsymbol{\theta}^*$ using that sample will constrain the particle set of $\boldsymbol{\theta}^*$ from properly adapting to the paths dictated by the future scenario if it is rather extreme. For example, if we consider a scenario on a conditioning variable that is six standard deviations away from its mean implied by $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$, the scenario will naturally push the parameter distribution away from $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ so as to be more compatible with the scenario. Hence the added costs of evaluating the data likelihood repeatedly may become necessary when the contemplated scenario is extreme to the point that sampling from $p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T})$ can no longer effectively boost the support.

A.4 Sample endpoints that fall in the target set

Recall that before we generate bridge paths for conditioning variable $y_{i,t}, i \in [1:k]$, we need to fix its endpoint $y_{i,(T+\tau)}$. For scenarios that require the endpoint to fall within a set, we will first sample $y_{i,(T+\tau)}$ according to some distribution denoted by g_i. If there is no reason for choosing a particular g_i, the truncated normal distribution based on the estimated AR(1) process will be a natural candidate when A_i is an interval. This density function only depends on $y_{i,T}$ and A_i, and we denote it by $g_i(y_{i,(T+\tau)}|y_{i,T}, A_i)$.

As an illustration, suppose the scenario requires $y_{i,(T+\tau)} \in A_i = [y_i^l, y_i^u]$. By assuming $y_{i,t}$ evolves according to equation (10), we sample $y_{i,(T+\tau)}$ from a normal distribution with the following mean and variance:

$$\mu_{i,T}(\tau) = E(y_{i,(T+\tau)}|y_{i,T}) = \rho_i^\tau y_{i,T} + \mu_i \frac{1-\rho_i^\tau}{1-\rho_i} \tag{27}$$

$$\sigma_{i,T}^2(\tau) = Var(y_{i,(T+\tau)}|y_{i,T}) = \frac{1-\rho_i^{2\tau}}{1-\rho_i^2}\sigma_i^2 \tag{28}$$

while restricting to the interval $[y_i^l, y_i^u]$.

The sampled endpoint $y_{i,(T+\tau)}$ based on truncated normality has the following density function:

$$\ln g_i(y_{i,(T+\tau)}|y_T, A_i) = \ln \phi\left(\xi_{i,T}(\tau)\right) - \ln \sigma_{i,T}(\tau) - \ln Z_{i,T}(\tau) \tag{29}$$

where

$$\begin{aligned}
\xi_{i,T}(\tau) &= \frac{y_{i,(T+\tau)} - \mu_{i,T}(\tau)}{\sigma_{i,T}(\tau)} \\
Z_{i,T}(\tau) &= \Phi\left(b_{i,T}(\tau)\right) - \Phi\left(a_{i,T}(\tau)\right) \\
a_{i,T}(\tau) &= \frac{y_i^u - \mu_{i,T}(\tau)}{\sigma_{i,T}(\tau)} \\
b_{i,T}(\tau) &= \frac{y_i^l - \mu_{i,T}(\tau)}{\sigma_{i,T}(\tau)},
\end{aligned}$$

$\phi(\cdot)$ is the density function of standard normal distribution and $\Phi(\cdot)$ is the corresponding cumulative distribution function. Note that $Z_{i,T}(\tau)$ need not be evaluated because it becomes an irrelevant constant as far as the importance weight is concerned.

A.5 Bridge-sampling degenerate state-space models

All linearized DSGE models can be expressed in a state-space form as in equations (7)-(8). Here we discuss the DSGE model of Smets and Wouters (2007), and show how our algorithm can deal with degeneracy in such a model. Notice that this model contains no measurement error and no constant term in the transition equation. Hence its state-space form looks as follows:

$$\begin{aligned}
\boldsymbol{s}_t &= \boldsymbol{B}\boldsymbol{s}_{t-1} + \boldsymbol{\Omega}\boldsymbol{\epsilon}_t^s \\
\boldsymbol{y}_t &= \boldsymbol{\mu} + \boldsymbol{M}\boldsymbol{s}_t \\
\boldsymbol{\epsilon}_t^s &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}^s)
\end{aligned}$$

where the vector of latent state variables, $\boldsymbol{s}_t$, is 50-dimensional, the vector of observable variables, $\boldsymbol{y}_t$, is seven-dimensional, $\boldsymbol{\epsilon}_t^s$ contains all Gaussian shock variables and is also seven-dimensional with covariance matrix $\boldsymbol{\Sigma}^s$. $\boldsymbol{\Omega}$ is a known 50×7 matrix containing the transformation information.

There are two types of degeneracy in the Smets and Wouters (2007) model. The first comes from dimension mismatch between the state variables and shocks, which is quite common in the state-space formulations converted from macro DSGE models. More precisely, the 50-dimensional state $\boldsymbol{s}_t$ is driven by a lower dimensional shock, i.e., seven-dimensional $\boldsymbol{\epsilon}_t^s$. Given $\boldsymbol{s}_{t-1}$, the variance of $\boldsymbol{s}_t$ equals $\boldsymbol{\Omega}\boldsymbol{\Sigma}^s\boldsymbol{\Omega}'$, which has a maximum rank of seven and hence is always singular. This makes it difficult to evaluate the density of $\boldsymbol{s}_t$. And since $\boldsymbol{\epsilon}_t^s$ and $\boldsymbol{s}_t$ carry essentially the same information, we can structure the augmented space in terms of $\boldsymbol{\epsilon}_t^s$ instead of $\boldsymbol{s}_t$.

The second type of degeneracy is associated with zero measurement error. In equation (12) for the initialization sampler, we have $\boldsymbol{\epsilon}_t^y = \boldsymbol{0}$, $\boldsymbol{\epsilon}_t^{y*} = \boldsymbol{M}\boldsymbol{\Omega}\boldsymbol{\epsilon}_t^s$, where $\boldsymbol{M}\boldsymbol{\Omega}$ is a 7×7 matrix and generically invertible. Thus, $f_G(\boldsymbol{\epsilon}_t^s|\boldsymbol{y}_t, \boldsymbol{s}_{t-1}, \boldsymbol{\theta})$ becomes a Dirac delta function with probability mass concentrated at a single point $\boldsymbol{\epsilon}_t^s = (\boldsymbol{M}\boldsymbol{\Omega})^{-1}(\boldsymbol{y}_t - \boldsymbol{\mu}^* - \boldsymbol{M}^*\boldsymbol{s}_{t-1})$. Denote in general the Dirac delta by $\delta_{\{\boldsymbol{x}=\boldsymbol{x}_0\}}(\boldsymbol{x})$ which is 0 everywhere except at $\boldsymbol{x}_0$ where its value equals infinity but the whole function can be integrated to 1. In short, $f_G(\boldsymbol{\epsilon}_t^s|\boldsymbol{y}_t, \boldsymbol{s}_{t-1}, \boldsymbol{\theta}) = \delta_{\{\boldsymbol{\epsilon}_t^s=(\boldsymbol{M}\boldsymbol{\Omega})^{-1}(\boldsymbol{y}_t-\boldsymbol{\mu}^*-\boldsymbol{M}^*\boldsymbol{s}_{t-1})\}}(\boldsymbol{\epsilon}_t^s)$ for such a degenerate case. For computing the importance weight, this Dirac delta function will be cancelled out by a same term in the target bridge density. But it is important to remember that this Dirac delta function depends on parameter value and bridge path so that cancellation can only happen if the target bridge's density shares the same Dirac delta function as the generating bridge.

The overall initialization density is then written as:

$$\begin{aligned} & \ln \mathcal{L}_{IBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \\ = \; & \ln I(\boldsymbol{\theta}) + \ln f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T}) + \ln \mathcal{L}_{IBridge}(\boldsymbol{Y}_{1:k}|\boldsymbol{y}_T) + \ln \mathcal{L}_{IBridge}(\boldsymbol{Y}_{k+1:n}|\boldsymbol{Y}_{1:k}, \boldsymbol{s}_T, \boldsymbol{\theta}) \\ & + \sum_{t=T+1}^{T+\tau} \ln \delta_{\{\boldsymbol{\epsilon}_t^s=(\boldsymbol{M}\boldsymbol{\Omega})^{-1}(\boldsymbol{y}_t-\boldsymbol{\mu}^*-\boldsymbol{M}^*\boldsymbol{s}_{t-1})\}}(\boldsymbol{\epsilon}_t^s) \end{aligned} \tag{30}$$

Note that the last term on the right-hand side of the above equation need not be evaluated due to a cancellation later, and for the coding purpose it can be set to 0.

Likewise for the target model,

$$\ln f_D(\boldsymbol{y}_t, \boldsymbol{\epsilon}_t^s|\boldsymbol{s}_{t-1}, \boldsymbol{\theta}) = \ln f_D(\boldsymbol{\epsilon}_t^s|\boldsymbol{s}_{t-1}, \boldsymbol{\theta}) + \ln f_D(\boldsymbol{y}_t|\boldsymbol{\epsilon}_t^s, \boldsymbol{s}_{t-1}, \boldsymbol{\theta})$$

and $f_D(\boldsymbol{y}_t|\boldsymbol{\epsilon}_t^s, \boldsymbol{s}_{t-1}, \boldsymbol{\theta})$ is again a Dirac delta function but in terms of $\boldsymbol{y}_t$. By equation (12) and $\boldsymbol{\epsilon}_t^y = \boldsymbol{0}$, we have

$$\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{M}\boldsymbol{C} + \boldsymbol{M}\boldsymbol{B}\boldsymbol{s}_{t-1} + \boldsymbol{M}\boldsymbol{\Omega}\boldsymbol{\epsilon}_t^s.$$

Factoring in the Jacobian of the transformation gives rise to

$$\begin{aligned} f_D(\boldsymbol{y}_t|\boldsymbol{\epsilon}_t^s, \boldsymbol{s}_{t-1}, \boldsymbol{\theta}) = & \frac{f_D(\boldsymbol{\epsilon}_t^s|\boldsymbol{y}_t, \boldsymbol{s}_{t-1}, \boldsymbol{\theta})}{det(\boldsymbol{M}\boldsymbol{\Omega})} \\ = & \frac{\delta_{\{\boldsymbol{\epsilon}_t^s=(\boldsymbol{M}\boldsymbol{\Omega})^{-1}(\boldsymbol{y}_t-\boldsymbol{\mu}^*-\boldsymbol{M}^*\boldsymbol{s}_{t-1})\}}(\boldsymbol{\epsilon}_t^s)}{det(\boldsymbol{M}\boldsymbol{\Omega})}. \end{aligned}$$

Hence, the log-density for the target bridge is

$$\begin{aligned}
& \ln \mathcal{L}_{TBridge}(\boldsymbol{Y}, \boldsymbol{E}_s, \boldsymbol{s}_T, \boldsymbol{\theta}) \\
\propto \;& \ln p(\boldsymbol{\theta}|\boldsymbol{y}_{1:T}) + \ln f_D(\boldsymbol{s}_T|\boldsymbol{\theta}, \boldsymbol{y}_{1:T}) + \sum_{t=T+1}^{T+\tau} \ln f_D(\boldsymbol{\epsilon}_t^s|\boldsymbol{s}_{t-1}, \boldsymbol{\theta}) - \tau \ln det(\boldsymbol{M\Omega}) \\
& + \sum_{t=T+1}^{T+\tau} \ln \delta_{\{\boldsymbol{\epsilon}_t^s = (\boldsymbol{M\Omega})^{-1}(\boldsymbol{y}_t - \boldsymbol{\mu}^* - \boldsymbol{M}^* \boldsymbol{s}_{t-1})\}}(\boldsymbol{\epsilon}_t^s) \qquad (31)
\end{aligned}$$

Note that the last term on the right-hand side of equation (31) is the same Dirac delta function as that in equation (30), again an irrelevant constant that can be set to 0 for the coding purpose.

After correctly evaluating the proposal density and target bridge density, the subsequent reweighting, resampling and boosting step follows exactly the same as described in A.2 - A.4.

B DSGE estimation results

Table 7: DSGE Posterior Distribution

Structural Parameters

	1966:1-2004:4		1966:1-2017:4	
	Mean	SD	Mean	SD
φ	5.74	0.96	4.65	0.86
σ_c	1.38	0.11	1.28	0.15
h	0.71	0.04	0.58	0.05
ξ_w	0.70	0.06	0.76	0.04
σ_l	1.83	0.55	1.76	0.48
ξ_p	0.66	0.05	0.76	0.04
ι_w	0.58	0.12	0.60	0.12
ι_p	0.24	0.08	0.29	0.09
ψ	0.54	0.11	0.79	0.07
Φ	1.60	0.08	1.51	0.07
r_π	2.04	0.17	2.03	0.15
ρ	0.81	0.02	0.82	0.02
r_y	0.08	0.02	0.09	0.02
$r_{\Delta y}$	0.22	0.03	0.24	0.02
$\bar{\pi}$	0.78	0.10	0.67	0.09
$\beta^{-1}-1$	0.16	0.05	0.16	0.05
$\bar{l}$	0.53	1.07	0.63	1.18
$\bar{\gamma}$	0.43	0.02	0.29	0.04
α	0.19	0.02	0.18	0.02

Shock Processes

	1966:1 - 2004:4		1966:1 - 2017:4	
	Mean	SD	Mean	SD
σ_a	0.45	0.03	0.46	0.03
σ_b	0.23	0.02	0.11	0.01
σ_g	0.53	0.03	0.48	0.02
σ_i	0.45	0.05	0.36	0.03
σ_r	0.24	0.02	0.22	0.01
σ_p	0.14	0.02	0.14	0.01
σ_w	0.24	0.02	0.38	0.02
ρ_a	0.95	0.01	0.99	0.004
ρ_b	0.22	0.08	0.79	0.04
ρ_g	0.97	0.01	0.97	0.01
ρ_i	0.71	0.05	0.78	0.06
ρ_r	0.15	0.06	0.16	0.05
ρ_p	0.89	0.05	0.87	0.05
ρ_w	0.96	0.01	0.98	0.01
μ_p	0.69	0.09	0.75	0.07
μ_w	0.84	0.05	0.96	0.01
ρ_{ga}	0.52	0.09	0.50	0.07

C Quality of the bridge-sampler

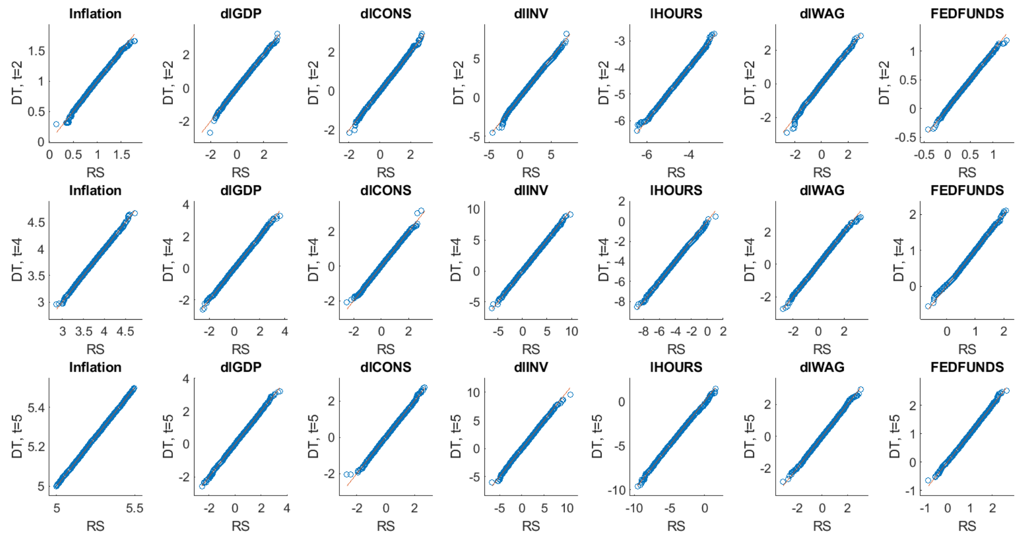


Figure 5: QQ plot for the VAR(1) model

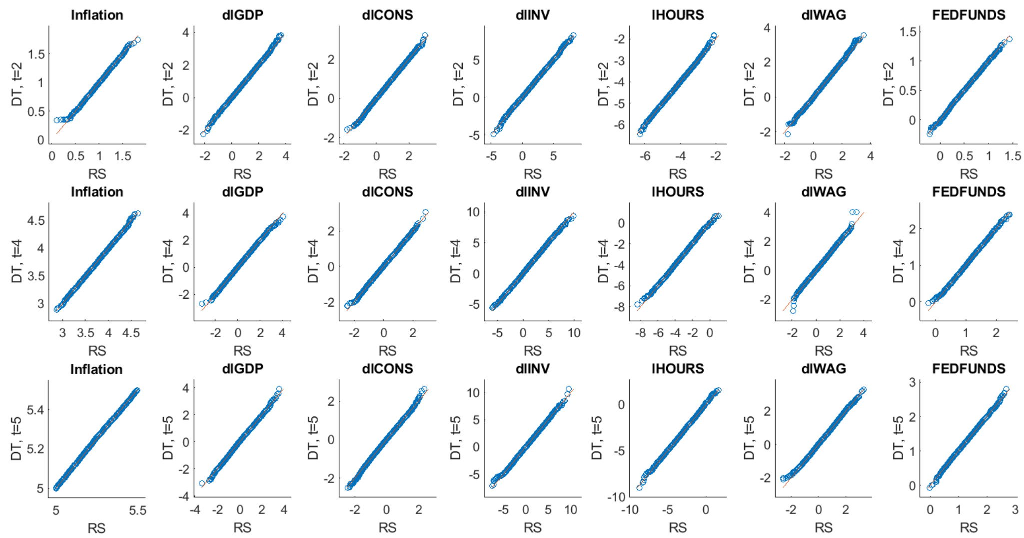


Figure 6: QQ plot for the DSGE model

D Out-of-sample forecast performance

Table 8: Out-of-Sample Forecast (2005:Q1 - 2017:Q4)

Forecast horizon	GDP	CONS	INV	Wage	*l*Hours	*dl*P	Fed funds	Ave-rage	Overall (SW)*
				VAR(1)					
1q	**0.63**	0.74	**1.77**	**1.06**	**0.45**	0.25	0.16	**0.72**	-11.08
2q	1.32	1.45	4.09	**1.14**	1.03	0.24	0.24	1.36	-6.32
4q	2.91	2.80	9.47	**1.33**	2.36	0.28	0.35	2.79	**-1.29**
8q	5.90	5.27	18.98	**1.86**	4.59	0.36	0.61	5.37	3.13
				BVAR(4)					
1q	0.64	0.68	1.87	1.11	0.49	**0.24**	0.11	0.74	**-11.73**
2q	**1.24**	1.31	3.91	1.22	**1.02**	**0.24**	**0.21**	**1.31**	**-6.42**
4q	**2.57**	2.45	**8.51**	1.47	**2.19**	**0.27**	**0.33**	**2.54**	-1.15
8q	**5.08**	4.59	**16.91**	2.08	**4.12**	0.32	**0.52**	**4.80**	3.67
				DSGE					
1q	0.71	**0.62**	1.79	1.11	0.68	0.25	**0.11**	0.75	-11.47
2q	1.44	**1.18**	**3.85**	1.26	1.41	0.28	0.22	1.38	-6.37
4q	2.96	**2.00**	8.57	1.77	2.81	0.27	0.42	2.69	-1.08
8q	5.66	**3.08**	17.39	3.31	5.00	**0.26**	0.72	5.06	**2.90**

Table 9: Out-of-Sample Forecast (2005:Q1 - 2009:Q2)

Forecast horizon	GDP	CONS	INV	Wage	*l*Hours	*dl*P	Fed funds	Ave-rage	Overall (SW)*
				VAR(1)					
1q	**0.81**	1.12	2.56	1.23	**0.58**	0.29	0.25	1.10	-11.32
2q	1.85	2.22	6.57	1.13	**1.52**	0.28	0.30	2.26	**-7.84**
4q	3.73	3.78	13.38	1.25	3.04	0.34	0.42	4.25	**-6.54**
8q	5.65	5.20	21.90	**1.92**	4.07	0.50	0.79	6.54	-7.75
				BVAR(4)					
1q	0.90	1.04	2.85	1.27	0.72	0.29	0.16	1.18	-11.22
2q	1.85	2.06	6.44	**1.11**	1.58	**0.28**	0.25	2.22	-7.50
4q	3.58	3.65	12.39	**1.24**	**2.90**	**0.28**	**0.41**	4.01	-5.89
8q	5.35	5.20	20.27	2.06	**3.70**	0.40	0.71	6.16	-7.67
				DSGE					
1q	0.83	**0.91**	**2.44**	**1.21**	0.85	**0.28**	**0.14**	**1.09**	**-12.32**
2q	**1.74**	**1.77**	**5.76**	1.16	1.77	0.32	**0.22**	**2.09**	-7.70
4q	**3.17**	**2.70**	**11.10**	1.43	2.95	0.28	0.44	**3.60**	-6.26
8q	**4.66**	**2.96**	**17.97**	2.50	3.91	**0.35**	**0.66**	**5.39**	**-8.55**

Table 10: Out-of-Sample Forecast (2009:Q3 - 2017:Q4)

Forecast horizon	GDP	CONS	INV	Wage	lHours	dlP	Fed funds	Ave-rage	Overall (SW)*
				VAR(1)					
1q	0.53	0.45	1.21	**0.97**	0.34	0.23	0.06	0.62	-15.43
2q	1.00	0.91	2.09	**0.97**	0.57	0.23	0.10	0.96	-11.43
4q	2.23	1.90	4.33	**1.33**	1.11	**0.21**	0.20	1.85	-7.40
8q	4.85	4.02	8.50	**1.69**	2.45	0.27	0.46	3.63	**-5.52**
				BVAR(4)					
1q	**0.46**	**0.35**	**1.08**	1.03	**0.31**	**0.21**	**0.05**	**0.57**	-16.04
2q	**0.80**	**0.68**	**1.76**	1.01	**0.53**	**0.22**	**0.10**	**0.83**	-11.76
4q	**1.74**	**1.42**	**3.47**	1.40	**0.88**	0.23	**0.16**	**1.52**	-7.43
8q	**3.75**	3.02	**6.21**	1.73	**1.78**	0.29	**0.32**	**2.80**	-4.83
				DSGE					
1q	0.66	0.40	1.37	1.06	0.53	0.23	0.10	0.71	**-16.08**
2q	1.30	0.79	2.55	1.16	1.03	0.25	0.21	1.18	**-12.31**
4q	2.80	1.47	5.64	1.96	2.05	0.23	0.43	2.36	**-8.84**
8q	5.51	**2.42**	11.61	3.78	3.81	**0.20**	0.75	4.56	**-8.37**

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