Term Structure and Bond Option Pricing under GARCH

Jin-Chuan Duan*

(First Draft: February 1996)
(Current Draft: October 1996)

Abstract

This model is based on a representative agent approach in an economy in which the nominal aggregate consumption growth rate follows the GARCH process. We first derive an equilibrium nominal interest rate process. This equilibrium nominal interest rate process is shown to be mean-reverting and heteroskedastic with a non-central χ^2 innovation. An equilibrium pricing measure is then constructed, and the equilibrium interest rate dynamics under this measure is characterized to serve as a pricing mechanism for the term structure of interest rates and interest rate derivatives. This new pricing approach is found to substantially differ from the term structure models of Vasicek (1977) and Cox, Ingersoll and Ross (1985) and the bond option pricing models of Jamshidian (1989) and Cox, Ingersoll and Ross (1985). It offers a promising alternative amid the negative empirical findings for the existing models.

Keywords: Nominal Interest Rate, Aggregate Consumption, Equilibrium Pricing Measure, GARCH Model, Term Structure, Bond Options, Monte Carlo Simulation.

*Correspondence to: J.C. Duan, Department of Finance, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong; Tel: (852) 2358-7671; Fax: (852) 2358-1749; E-mail: jcduan@ust.hk; Web: http://www.bm.ust.hk/~jcduan. The first draft of this paper was completed while the author was at McGill University. The funding supports by McGill University and the Social Sciences and Humanities Research Council of Canada are acknowledged.

1 Introduction

In a recent article, Duan (1995) developed an option pricing model based on the premise that asset returns follow the GARCH process. The motivations for the GARCH option pricing model are two-fold. First, extensive empirical evidence has strongly indicated that the GARCH model can better describe the heteroskedastic and leptokurtic properties exhibited by asset returns; see for example, Bollerslev, Chou and Kroner (1992). Second, the Black-Scholes (1973) option pricing model is known to exhibit systematic biases. It is shown in Duan (1995) that the GARCH option pricing model is able to explain away the systematic biases exhibited by the Black-Scholes (1973) model. Empirically, Amin and Ng (1993) tested the GARCH option pricing model using three different GARCH specifications: linear GARCH of Bollerslev (1986), exponential GARCH of Nelson (1991) and the GJR-GARCH of Glosten, Jagannathan and Runkle (1993). They concluded that the GARCH option pricing model performs significantly better than the Black-Scholes (1973) model. Heynen, Kemna and Vorst (1994), on the other hand, used the GARCH option pricing model to examine the term structure of implied volatilities. They concluded that among the bivariate diffusion option pricing model of Hull and White (1987) and the two versions of the GARCH option pricing model, the exponential GARCH version is the only one not being rejected. In a study of using the GARCH option pricing model to describe "volatility smile", Duan (1996a) has found the volatility smile and term structure of implied volatilities of stock index options can be simultaneously fitted to a high degree of accuracy, and the out-of-sample fitting performance remains excellent.

In this paper, we generalize the pricing methodology of Duan (1995) to cover interest rate derivative contracts. Our model development takes as the first step the derivation of equilibrium nominal interest rate. We consider a representative agent economy in which the aggregate consumption growth rate follows the GARCH process. Our derivation shows that the equilibrium nominal interest rate is a linear transformation of the conditional variance, and as a result, the equilibrium nominal interest rate dynamics is a mean-reverting, heteroskedastic process with a non-central χ^2 innovation.[1] The second step of our derivation is to devise an equilibrium pricing measure which can be used to facilitate the development of pricing models for bonds and options. Under the equilibrium pricing measure, the dynamics of the equilibrium nominal interest rate is altered somewhat. The innovation undergoes a mean shift, but again follows a non-central χ^2 distribution with a different non-centrality parameter. This derived dynamics makes it possible to price bonds and bond options under the GARCH setting.

We numerically examine the properties of the term structure and bond option pricing models. For the term structure of interest rates, the GARCH-based approach is compared to both the Vasicek (1977) and Cox, Ingersoll and Ross (CIR) (1985) models. The strategy is to first generate

[1] This endogenously derived interest rate process differs from the constant elasticity of variance (CEV) or GARCH specification that is sometimes adopted by researchers. For the examples of the CEV specification, see Vasicek (1977), Brennan and Schwartz (1980), Courtadon (1982) and Cox, Ingersoll and Ross (1985). The examples of using the GARCH model can be found in Weiss (1984) and Bollerslev, Engle and Wooldridge (1988).

a normal yield curve based on the GARCH term structure model, and then to fit the Vasicek and CIR term structure to this generated yield curve. For the Vasicek model, it is possible to adjust parameter values to achieve a very good fit. It is not so for the CIR model with the fitted yields to be higher than the target yields for shorter maturities and lower for longer maturities. If one increases or decreases the short-term rate without changing the parameter values, the yield curves corresponding to three models begin to differ even more. Using the same fitted parameter values, the GARCH-based bond option pricing model is also compared to the model of Jamshidian (1989) and that of CIR. The GARCH-based model always yields a higher option price in comparison to the CIR model. It also produces higher prices, as compared to the Jamshidian model (1989), except for deep out-of-the-money bond options. Given that the models by Vasicek (1977), CIR (1985) and Jamshidian (1989) are known to have unsatisfactory empirical performance, the GARCH-based approach offers a promising alternative.

2 Equilibrium nominal interest rate dynamic

Let C_t and c_t denote the nominal and real aggregate consumptions at time t, respectively. The price deflator at time t is denoted by π_t. By definition, $C_t = c_t\pi_t$. We assume that the nominal aggregate consumption growth rate (continuously compounded) follows a GARCH-Mean model of Engle, Lilien and Robins (1987). Specifically, we adopt the non-linear asymmetric GARCH (NGARCH) process of order $(1,1)$ (see Engle and Ng, 1993).[2] If the parameter ψ, to be specified below, equals zero, then the model reduces to a linear GARCH model. Let $\mathcal{F}_t$ denote the information set up to and including time t, and P denote the data generating probability measure that governs all random variables. A formal description of our assumption is now in order.

Assumption 1 Under the data generating probability measure P,

$$\triangle ln(C_t) = \mu + \kappa q_{t+1} + \sqrt{q_{t+1}}\varepsilon_{t+1} \quad (1)$$

$$\varepsilon_{t+1}|\mathcal{F}_t \sim N(0,1) \quad (2)$$

$$q_{t+1} = \alpha_0 + \alpha_1 q_t(\varepsilon_t - \psi)^2 + \alpha_2 q_t \quad (3)$$

where $\alpha_0 > 0, \alpha_1 \geq 0, \alpha_2 \geq 0$ and $\alpha_1(1+\psi^2) + \alpha_2 < 1$.

The parameter restriction that $\alpha_0 > 0, \alpha_1 \geq 0$ and $\alpha_2 \geq 0$ ensures positive conditional variances. According to Duan (1996b, Corollary 2), the restriction that $\alpha_1(1+\psi^2) + \alpha_2 < 1$ ensures the existence of stationary variance and the strict stationarity for the conditional variance process.

We assume a representative agent economy in which the agent has an additive time-separable preference. The utility function of real consumption, c_t, is denoted by $U(c_t)$. For any nominal asset

[2] We use the NGARCH(1,1) model because of its simplicity and its ability to capture several key features, such as time-varying and clustered volatilities, fat-tailed distributions and leverage effect, exhibited by financial time series. All theoretical results derived in this paper can be further generalized to the augmented GARCH(1,1) setting of Duan (1996b). The theoretical arguments remain the same and the resulting equilibrium interest rate dynamics continues to be Markovian.

with the nominal price, X_t, and the dividend payment, D_t, at time t, the following Euler's equation must be satisfied in equilibrium:

$$\frac{X_t}{\pi_t} = E^P\{e^{-\delta}\frac{U'(c_{t+1})}{U'(c_t)}\frac{(X_{t+1}+D_{t+1})}{\pi_{t+1}}|\mathcal{F}_t\} \tag{4}$$

where δ is an intertemporal discount rate. The model of this nature has been widely employed in the literature. Lucas (1978), Mehra and Prescott (1985) and Campbell (1986) are some examples. In contrast to most existing models, our model is unique in the sense that the nominal aggregate consumption growth rate follows an NGARCH(1,1) process. The existence of equilibrium for this model is essentially that of Lucas (1978).[3] In this article, we are not concerned with the issue of existence. Instead, Euler's equation in (4) serves as the basis for our derivations.

Let r_t denote the nominal one-period risk-free interest rate (continuously compounded) at time t. Consider a one-dollar payoff at time $t+1$. Its time-t nominal value equals $exp(-r_t)$ by definition, and the use of (4) leads to the following relationship:

$$e^{-r_t} = E^P\{e^{-\delta}\frac{U'(c_{t+1})}{U'(c_t)}\frac{\pi_t}{\pi_{t+1}}|\mathcal{F}_t\}. \tag{5}$$

In order to characterize the dynamics for r_t, the representative agent must be endowed with a more specific preference structure. We adopt the logarithmic utility function, which is an assumption frequently employed in the term structure literature. This assumption allows for combining the real consumption with the price deflator, which turns the nominal consumption into a sufficient statistic for asset pricing.

Assumption 2 $U(c_t) = ln(c_t)$

The following theorem completely characterizes the dynamics for the equilibrium nominal interest rate.

Theorem 1 Under the data generating probability measure P,

1.

$$\triangle r_t = \phi(m - r_t) + \sigma(r_t - a)W_{t+1} \tag{6}$$

where

$$\begin{aligned}
\phi &= 1 - \alpha_1(1+\psi^2) - \alpha_2, a = \delta + \mu, \\
\sigma &= \alpha_1\sqrt{2+4\psi^2}, m = a + \frac{(2\kappa-1)\alpha_0}{2\phi}, \\
W_{t+1} &= \frac{(\varepsilon_{t+1}-\psi)^2 - (1+\psi^2)}{\sqrt{2+4\psi^2}}, \\
E^P(W_{t+1}|\mathcal{F}_t) &= 0, \text{ and } E^P(W_{t+1}^2|\mathcal{F}_t) = 1
\end{aligned}$$

[3] In Lucas (1978), the existence and uniqueness of the exchange equilibrium were established under the assumption of a Markovian output. His proof hinges on treating the asset price as a function of the output. In the current setting, the aggregate consumption is the aggregate output in equilibrium if the equilibrium exists. Since the aggregate consumption follows the GARCH(1,1) process, it is not Markovian in a univariate sense. The Lucas arguments continue to apply to this case, however, because the GARCH(1,1) process can be made into a bivariate Markovian process (see Duan, 1995).

2. a is a lower (upper) bound of nominal interest rates if $\kappa > (<)\frac{1}{2}$, and $r_t = a$ if $\kappa = \frac{1}{2}$;

3. r_t is a strictly stationary process.

Proof: See Appendix.

This endogenous characterization of the interest rate dynamics differs from the usual constant elasticity of variance (CEV) specification in two aspects. First, the conditional standard deviation described in (6) is a linear function of the difference between the prevailing interest rate and a constant. In the CEV specification, this constant is set equal to zero. The linear specification for the conditional volatility is also more consistent with some empirical findings; for example, Chan, et al. (1992), Rumsey (1993) and Honore (1996). These findings suggest the estimated CEV coefficient is more in line with linearity. Second, in contrast to the use of a Gaussian innovation, the interest rate process in (6) is driven by a non-central χ^2 random variable with its degrees of freedom equal to one. Comparing to the GARCH specification adopted by many to model the interest rate process, the dynamic in (6) implies that the interest rate process resembles more the conditional variance process of a GARCH model than a GARCH process itself. This is because the equilibrium interest rate is a linear transformation of the conditional variance. In other words, the equilibrium interest rate dynamics is essentially a process for the conditional variance of the aggregate nominal consumption growth rate. This link has also been utilized in a recent paper of Duan and Jocobs (1996) to reconcile two seemingly unrelated empirical findings on the long-memory properties of index returns and interest rates.

3 Equilibrium pricing measure

For asset pricing, it is often more convenient to create a new probability measure that is different from the data generating probability measure. The ideal candidate is the one supportive of the standard martingale pricing result. Toward this aim, we define a measure Q by

$$dQ = e^{-\delta T + \sum_{i=0}^{T-1} r_i} \frac{U'(c_T)}{U'(c_0)} \frac{\pi_0}{\pi_T} dP \tag{7}$$

for some large finite T. Since $\int 1 dQ = 1$ and the Radon-Nykodyn derivative in (7) is positive, Q is a probability measure. Since measure Q is derived by employing marginal rates of substitution, Q is hereafter referred to as the equilibrium pricing measure. Under Assumption 2, the defining relationship in (7) becomes

$$dQ = e^{-\delta T + \sum_{i=0}^{T-1} r_i} \frac{C_0}{C_T} dP \tag{8}$$

The following theorem provides a key linkage relating the conditional expectation operations under two different probability measures.

Theorem 2 For any $\mathcal{F}_{t+1}$-measurable function, Z_{t+1}, $t \leq T-1$,

$$E^Q(Z_{t+1}|\mathcal{F}_t) = E^P(Z_{t+1} e^{r_t - \delta} \frac{C_t}{C_{t+1}} |\mathcal{F}_t) \tag{9}$$

Proof: See Appendix.

With a change in measures, Euler's equation in (4) can be stated in a different way. Corollary 1 establishes a result that is consistent with the well-known martingale pricing principle.

Corollary 1 For $t \leq T-1$,

$$X_t = E^Q\{e^{-r_t}(X_{t+1} + D_{t+1})|\mathcal{F}_t\} \tag{10}$$

Proof: Follows from equation (4), Assumption 2 and Theorem 2.

Although the result in Corollary 1 serves as the conceptual basis for deriving the pricing models for bonds and bond options, it is not an operational model unless the complete knowledge of the distributional characteristics, under measure Q, of the equilibrium interest rate is known to us.

4 Term structure and bond option pricing

In this section, our task is to characterize the equilibrium interest rate under the equilibrium pricing measure Q, which in turn yields the pricing models for bonds and bond options. The dynamics of the equilibrium interest rate in Theorem 1 is expected to obey a different stochastic process under the equilibrium pricing measure Q. Define a notation $\eta \equiv \frac{\alpha_0}{(m-a)\phi}$. The following theorem provides a complete characterization of the equilibrium interest rate's dynamics under Q.

Theorem 3 Under the equilibrium pricing measure Q, for $t \leq T-1$,

$$\begin{aligned} \Delta r_t &= \phi(m-r_t) + \frac{\sigma\eta}{\sqrt{2+4\psi^2}}(r_t-a)^2 + \frac{2\sigma\psi\sqrt{\eta(r_t-a)}}{\sqrt{2+4\psi^2}}(r_t-a) + \\ &\quad \sigma(r_t-a)\sqrt{1+\frac{2\eta(r_t-a)+4\psi\sqrt{\eta(r_t-a)}}{1+2\psi^2}}W^*_{t+1} \end{aligned} \tag{11}$$

where

$$\begin{aligned} \varepsilon^*_{t+1} &= \varepsilon_{t+1} + \sqrt{\eta(r_t-a)} \sim N(0,1) \text{ conditional on } \mathcal{F}_t, \\ W^*_{t+1} &= \frac{[\varepsilon^*_{t+1} - \psi - \sqrt{\eta(r_t-a)}]^2 - [1 + (\psi + \sqrt{\eta(r_t-a)})^2]}{\sqrt{2+4(\psi+\sqrt{\eta(r_t-a)})^2}}, \\ E^Q(W^*_{t+1}|\mathcal{F}_t) &= 0, \text{ and } E^Q(W^{*2}_{t+1}|\mathcal{F}_t) = 1 \end{aligned}$$

Proof: See Appendix.

The parameter η is positive (negative) if $m > (<) a$. If a is a lower (upper) bound, then $m > (<) a$. In either case, $\eta(r_t - a)$ is always positive, and hence the dynamic in (11) is well-defined. According to Theorem 3, the interest rate process under Q is driven by a non-central χ^2 innovation whose degrees of freedom equals one and non-centrality parameter is a function of the interest rate.

This theorem indicates that the conditional non-centrality parameter is $\psi + \sqrt{\eta(r_t - a)}$.[4] This non-centrality parameter reflects the magnitude of the prevailing interest rate in relation to the parameter a. The dynamics in (11) contains six parameters: ϕ, m, σ, a, ψ and η. In contrast to the dynamics in (6), the additional parameter η can be likened to the risk premium parameter in the term structure models such as Vasicek (1977) and CIR (1985).

Using Corollary 1 and the law of iterated expectations, the time-t price for per dollar face value of a zero-coupon, default-free bond with maturity τ, denoted by $B_t(\tau)$, can be written as

$$B_t(\tau) = E^Q(e^{-\sum_{i=t}^{t+\tau-1} r_i} | \mathcal{F}_t). \tag{12}$$

Let $R_t(\tau)$ denote the yield to maturity at time t for a default-free, zero-coupon bond of maturity τ. Then,

$$R_t(\tau) = -\frac{1}{\tau} ln\{E^Q(e^{-\sum_{i=t}^{t+\tau-1} r_i} | \mathcal{F}_t)\}. \tag{13}$$

For a coupon bond, its value can be simply computed with repeated use of (12) to calculate the values for all future payments associated with this coupon bond. The pricing formula in (12) thus represents an equilibrium single-factor term structure model. The model in (12) has no closed-form solution except for the case that $\tau = 2$. The closed-form solution for this case can be derived by applying the moment generating function for the non-central χ^2 distribution with degrees of freedom 1. In general, this term structure model must be computed using some numerical methods. In this paper, we use Monte Carlo simulations to compute bond prices.

For the pricing of bond options, the valuation equation also follows from Corollary 1 and the law of iterated expectations. For example, consider a European-style call option, maturing at time T and with exercise price K, written on a zero-coupon bond with maturity τ and face value F. Denote its time-t value by O_t^B. It follows that

$$O_t^B = E^Q\{e^{-\sum_{i=t}^{T-1} r_i} Max[B_T(\tau)F - K, 0] | \mathcal{F}_t\} \tag{14}$$

For options on a coupon bond, simply replace $B_T(\tau)$ in the above equation with a suitable coupon bond counterpart. The computation of this model must again be based on some numerical methods. In this paper, we use Monte Carlo simulations to compute the option value. Since the bond price at the expiration of the option must also be computed by Monte Carlo simulations, we have come up with a quick procedure that makes the simulation task manageable. This procedure calls for storing a set of bond prices in memory. This simulation procedure will be discussed in the next section.

5 Numerical examples

The GARCH-based models for term structure of interest rates and bond options are demonstrated numerically in this section. Comparisons to some existing models are made in order to gain further

[4]The non-centrality parameter of a non-central χ^2 is usually defined as the square of the mean-shift parameter. For expositional convenience, we ignore the squaring operation.

insights into the properties of the GARCH-based models. The GARCH-based term structure model and the models by Vasicek (1977), CIR (1985) are non-nested. To make comparisons, we act as if all three models are fitted to a given yield curve. In this section, we choose the GARCH-based term structure model as the benchmark. Since the purpose of this comparison is to examine the differences among these three models, we could actually use any one of the three models to generate the target yield curve.

For simplicity, we assume the basic unit of time is one week. Therefore, the shortest-term interest rate is of one week. The parameter values used in these comparisons are $m = 0.001154, \phi = 0.2, \sigma = 0.3, a = 0, \psi = 0, \eta = 100$. The value for m yields 6% (annual rate) for the long-run average one-week interest rate. As shown later in this section, the GARCH-based term structure model with these parameter values will yield a long-term interest rate around 11%.

5.1 Term structure of interest rates

We use the GARCH-based model in (13) to produce the yield curves corresponding to the low, average and high one-week interest rates. The bond pricing model is computed using Monte Carlo simulations with 50,000 sample paths. Each of these three yield curves is compared to the models by Vasicek (1977) and CIR (1985). The parameters for these two comparison models are estimated using 520 yields for the maturity ranging from 1 to 520 weeks. These yields are obtained by using the GARCH-based term structure model and setting the current one-week interest rate to its long-run average value, m. The parameter estimation is carried out as a cross-sectional non-linear regression forcing the non-linear system to be either the Vasicek or CIR model. Since the use of a non-linear cross-sectional regression to estimate these two models is known to have an under-identification problem, we choose to fix their respective long-run average one-week interest rates at the value for m, which equals 0.001154. The rest of parameters in the Vasicek (1977) model are estimated to be 0.023614 for the mean-reversion intensity, 0.001308 for the volatility parameter and 0.044873 for the risk premium. For the CIR (1985) model, the estimated parameter values are 0.519632, 0.919471 and -1.655329, respectively.

Figure 1a plots the yield (annualized) curve, up to 300 weeks, corresponding to the GARCH-based term structure model and the fitted curves for the Vasicek and CIR models. As the result indicates, the fit for the Vasicek model is extremely accurate, whereas the fit for the CIR model is quite poor, too high for shorter maturities and low for longer maturities. The inability of the CIR model to fit the yield curve generated by the GARCH-based term structure model reflects the fundamental difference in the curvatures exhibited by these two models.

[Enter Figure 1a, 1b and 1c here]

The difference among the GARCH-based term structure model and the two comparison models becomes very pronounced in the time-series dimension. Since the short-term interest rate will change over time, it is important to know how these yield curves respond to the change. If we let the one-week interest rate to be half of its long-run average, the yield curves for these three models depart significantly. The CIR model produces even larger yields, relative to the GARCH-based model. The Vasicek model, on the other hand, gives rise to lower yields over the entire maturity spectrum than the GARCH-based model (see Figure 1b). If the one-week interest rate becomes twice as large as its long-run average, the relationship is reversed. Figure 1c indicates that the CIR model produces lower yields than the GARCH-based model except for the short end of the maturity spectrum. Interestingly, the Vasicek model produces higher yields throughout. In summary, the direction of departure from the GARCH-based model depends on the position of the one-week rate relative to its long-run average.

5.2 Bond options

Options to buy the three-month Treasury bill are used as the example of bond options. The analysis is conducted for a range of moneyness positions and interest rate levels. The model in (14) is calculated by Monte Carlo simulations. Since the price for the three-month Treasury bill at the option's expiration must also be computed using Monte Carlo simulations, the overall Monte Carlo simulations could be carried out in a tree-like structure with one intermediate node. This way of computing, however, presents a numerical difficulty; for example, if we simulate 50,000 bond prices at the option's maturity and each of the bond prices is computed by 1,000 interest rate paths, the total number of simulated sample paths equals $5\text{x}10^7$.

To overcome this difficulty, we have devised a particular simulation scheme. First, we compute and store in memory the bond prices corresponding to 300 different interest rates with each being computed using 1,000 interest rate paths. Since the system is Markovian, the only stochastic element of the bond price is the interest rate at the option's maturity. These interest rates (after annualizing) ranges from 10 basis points to 3,000 basis points with the increment of 10 basis points. Second, we simulate 50,000 sample paths of the short-term interest rate all the way to the expiration point of the option. To compute the option's payoff, we pick one of the pre-computed bond prices that corresponds to the interest rate nearest to the simulated ending interest rate. Finally, we compute the sample equivalent of the quantity in equation (14) by averaging the discounted payoffs. This Monte Carlo scheme requires 350,000 simulated sample paths with 300,000 for the underlying bond and 50,000 for the option. Of course, any satisfactory level of accuracy can be achieved by adjusting the number of bond prices and the range of their corresponding interest rates. Since the bond prices stored in memory are independent of the exercise price and maturity of the option, they can be repeatedly used for options with different exercise prices and maturities, a very substantial saving in computing time indeed.

Two alternative bond option pricing models - Jamshidian (1989) and CIR (1985) - are used to compare with the GARCH bond option pricing model. Since Jamshidian's model uses the specification of Vasicek (1977) for interest rates, the parameter values can be taken from the preceding sub-section. For the CIR bond option pricing model, the parameter values are also available from the preceding sub-section.

For Figures 2a, 2b and 2c, we assume that the face value of the underlying three-month Treasury bill equals 1. The face value to exercise price ratio (F/K) ranges from 1 to 1.5. The maturity of all options is fixed at 13 weeks. Interestingly, Jamshidian's model in all cases yields a strictly positive option value when $F/K = 1$. Since the market value of a zero-coupon bond cannot exceed its face value, the option payoff for the case that $F/K = 1$ must be zero always. Jamshidian's model violates this rational option pricing bound due to the assumption of Gaussian interest rates. In other words, the possibility of having negative interest rates leads to the occurrence of the situation in which the zero-coupon bond price is higher than its face value. This problem does not exist for either the CIR model or the GARCH-based model because interest rates in these two models are always positive.

[Enter Figure 2a, 2b and 2c here]

In Figure 2a, the one-week interest rate is set equal to its long-run average. Jamshidian's model clearly yields the highest option prices up to some moneyness position. This again is due to its allowance for negative interest rates. The relationship between the GARCH-based model and the CIR model is one-sided with the CIR model yielding lower prices. For Figures 2b and 2c, the one-week interest rate is set at half and twice of its long-run average, respectively. Comparing to the CIR model, the GARCH-based model continues to produce higher values in either case. The relationship between the Jamshidian model and the GARCH-based model remains, in either case, essentially the same as that in Figure 2a. The crossing point, however, moves to the right in the case of a lower one-week interest rate, and to the left in the case of a higher interest rate, clearly due to the change in the conditional probability of obtaining negative interest rates. In summary, the GARCH-based model yields higher bond option values in comparison to the Jamshidian and CIR models, with the exception of the negative interest rate effect permissible under the Jamshidian model.

6 Conclusion

A representative agent economy is used to develop a model that can deal with the pricing of bonds and interest rate derivatives in a GARCH framework. A stochastic equilibrium nominal short-term interest rate process is derived endogenously. The equilibrium short-term interest rate is shown to follow a process different from the usual specifications found in the literature. In equilibrium, the interest rate dynamics is a mean-reverting, heteroskedastic process with a non-central χ^2 innovation. Using the devise of the equilibrium pricing measure, the pricing models for bonds and bond options are derived. Numerical comparisons suggest that this new modeling approach yields results substantially different from the existing term structure and bond option pricing models. Since the existing models generally perform rather unsatisfactorily in empirical studies, there is a need for alternative models. The GARCH-based approach offers such an alternative, but needs future research to determine its empirical performance.

7 Appendix

Proof of Theorem 1

Substituting $U(c_t) = ln(c_t)$ into (5) yields

$$e^{-r_t} = E^P[e^{-\delta}\frac{c_t}{c_{t+1}}\frac{\pi_t}{\pi_{t+1}}|\mathcal{F}_t]$$

$$
\begin{aligned}
&= E^P[e^{-\delta}\frac{C_t}{C_{t+1}}|\mathcal{F}_t] \\
&= e^{-\delta}E^P[e^{-\mu-\kappa q_{t+1}-\sqrt{q_{t+1}}\varepsilon_{t+1}}|\mathcal{F}_t] \\
&= e^{-(\delta+\mu)-\frac{2\kappa-1}{2}q_{t+1}}
\end{aligned}
$$

Thus, $r_t = \delta + \mu + \frac{2\kappa-1}{2}q_{t+1}$. It is clear that r_t is a constant if $\kappa = \frac{1}{2}$. Since $q_{t+1} > 0$, it follows that $r_t > (<)\delta + \mu$ if $\kappa > (<)\frac{1}{2}$. This completes the proof for item (2). Since q_{t+1} is a stationary process by Assumption 1 (see Duan, 1996, Corollary 2), r_t must also be a stationary process by virtue of the above relationship. Item 3 is thus established. Substituting the relationship between r_t and q_{t+1} into the variance equation in (3) yields

$$
\begin{aligned}
r_{t+1} - \delta - \mu &= \frac{2\kappa-1}{2}\alpha_0 + [\alpha_1(\varepsilon_{t+1}-\psi)^2 + \alpha_2](r_t - \delta - \mu) \\
&= \frac{2\kappa-1}{2}\alpha_0 + [\alpha_1(1+\psi^2) + \alpha_2](r_t - \delta - \mu) + \\
&\quad \alpha_1\sqrt{2+4\psi^2}(r_t - \delta - \mu)\frac{(\varepsilon_{t+1}-\psi)^2 - (1+\psi^2)}{\sqrt{2+4\psi^2}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
r_{t+1} &= \frac{2\kappa-1}{2}\alpha_0 + \phi a + [\alpha_1(1+\psi^2)+\alpha_2]r_t + \sigma(r_t - a)W_{t+1}, \text{ or} \\
\triangle r_t &= \frac{2\kappa-1}{2}\alpha_0 + \phi a - \phi r_t + \sigma(r_t - a)W_{t+1} \\
&= \phi(m - r_t) + \sigma(r_t - a)W_{t+1}
\end{aligned}
$$

The proof is now complete.□

Proof of Theorem 2

Consider any bounded $\mathcal{F}_t$ - measurable function, ϕ_t.

$$E^Q(Z_{t+1}\phi_t) = E^Q[\phi_t E^Q(Z_{t+1}|\mathcal{F}_t)].$$

Alternatively, by (7)

$$
\begin{aligned}
E^Q(Z_{t+1}\phi_t) &= E^P(\phi_t Z_{t+1}e^{-\delta T+\sum_{i=0}^{T-1} r_i}\frac{C_0}{C_T}) \\
&= E^P[\phi_t e^{-\delta t+\sum_{i=0}^{t-1} r_i}\frac{C_0}{C_t}E^P(Z_{t+1}e^{-\delta(T-t)+\sum_{i=t}^{T-1} r_i}\frac{C_t}{C_T}|\mathcal{F}_t)]
\end{aligned}
$$

Let $\psi_t \equiv E^P(e^{-\delta(T-t)+\sum_{i=t}^{T-1} r_i}\frac{C_t}{C_T}|F_t)$. It follows from the law of iterated expectations that $\psi_t = 1$. The preceding equation can be further deduced to yield

$$
\begin{aligned}
E^Q(Z_{t+1}\phi_t) &= E^P[\phi_t e^{-\delta t+\sum_{i=0}^{t-1} r_i}\frac{C_0}{C_t}\psi_t E^P(Z_{t+1}e^{-\delta(T-t)+\sum_{i=t}^{T-1} r_i}\frac{C_t}{C_T}|\mathcal{F}_t)] \\
&= E^P[\phi_t e^{-\delta T+\sum_{i=0}^{T-1} r_i}\frac{C_0}{C_T}E^P(Z_{t+1}e^{-\delta(T-t)+\sum_{i=t}^{T-1} r_i}\frac{C_t}{C_T}|\mathcal{F}_t)] \\
&= E^Q[\phi_t E^P(Z_{t+1}e^{-\delta(T-t)+\sum_{i=t}^{T-1} r_i}\frac{C_t}{C_T}|\mathcal{F}_t)]
\end{aligned}
$$

Thus, $E^Q(Z_{t+1}|\mathcal{F}_t) = E^P(Z_{t+1}e^{-\delta(T-t)+\sum_{i=t}^{T-1} r_i}\frac{C_t}{C_T}|\mathcal{F}_t)$. Applying the law of iterated expectations, the statement of this theorem is established.□

Proof of Theorem 3

We first characterize the distribution for ε_{t+1} under measure Q.

$$\begin{aligned}
E^Q(e^{\omega\varepsilon_{t+1}}|\mathcal{F}_t) &= E^P(e^{\omega\varepsilon_{t+1}+r_t-\delta}\frac{C_t}{C_{t+1}}|\mathcal{F}_t) \\
&= e^{r_t-a-\kappa q_{t+1}}E^P[e^{(\omega-\sqrt{q_{t+1}})\varepsilon_{t+1}}|\mathcal{F}_t] \\
&= \exp(r_t - a - \kappa q_{t+1} + \frac{\omega^2}{2} - \omega\sqrt{q_{t+1}} + \frac{q_{t+1}}{2})
\end{aligned}$$

The first equality is due to Theorem 2, whereas the second and third equalities follow from Assumption 1.

Let $\omega = 0$. This implies that $\exp(r_t - a - \kappa q_{t+1} + \frac{q_{t+1}}{2}) = 1$, and thus,

$$E^Q(e^{\omega\varepsilon_{t+1}}|\mathcal{F}_t) = \exp(-\omega\sqrt{q_{t+1}} + \frac{\omega^2}{2}).$$

By the property of moment generating function, the following must be true:

$$\varepsilon_{t+1}|\mathcal{F}_t \overset{Q}{\sim} N(-\sqrt{q_{t+1}}, 1).$$

From the proof for Theorem 1, $r_t = a + \frac{2\kappa-1}{2}q_{t+1} = a + \frac{q_{t+1}}{\eta}$. It follows that

$$\begin{aligned}
&\varepsilon_{t+1}|\mathcal{F}_t \overset{Q}{\sim} N(-\sqrt{\eta(r_t-a)}, 1), \text{ or} \\
&\varepsilon^*_{t+1}|\mathcal{F}_t \overset{Q}{\sim} N(0, 1).
\end{aligned}$$

It is now straightforward to verify that W^*_{t+1}, defined in the theorem, has a zero conditional mean and a unit conditional variance. Use of equation (6) yields

$$\begin{aligned}
\Delta r_t &= \phi(m - r_t) + \sigma(r_t - a)\frac{(\varepsilon_{t+1} - \psi)^2 - (1+\psi^2)}{\sqrt{2+4\psi^2}} \\
&= \phi(m - r_t) + \sigma(r_t - a)\frac{(\varepsilon^*_{t+1} - \psi - \sqrt{\eta(r_t-a)})^2 - (1+\psi^2)}{\sqrt{2+4\psi^2}} \\
&= \phi(m - r_t) + \frac{\sigma(r_t - a)}{\sqrt{2+4\psi^2}}\left[\sqrt{2+4(\psi+\sqrt{\eta(r_t-a)})^2}W^*_{t+1} + \eta(r_t-a) + 2\psi\sqrt{\eta(r_t-a)}\right] \\
&= \phi(m - r_t) + \frac{\sigma\eta}{\sqrt{2+4\psi^2}}(r_t-a)^2 + \frac{2\sigma\psi\sqrt{\eta(r_t-a)}}{\sqrt{2+4\psi^2}}(r_t-a) + \\
&\quad \sigma(r_t-a)\sqrt{1+\frac{2\eta(r_t-a)+4\psi\sqrt{\eta(r_t-a)}}{1+2\psi^2}}W^*_{t+1}
\end{aligned}$$

The dynamics in (11) is now established.□

References

[1] Amin, K. and V. Ng, 1993, ARCH Processes and Option Valuation, University of Michigan working paper.

[2] Black, F. and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–659.

[3] Bollerslev, T., 1986, Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics* 31, 307–327.

[4] Bollerslev, T., R. Chou and K. Kroner, 1992, ARCH Modeling in Finance: A Review of the Theory and Empirical Evidence," *Journal of Econometrics* 52, 5-59.

[5] Bollerslev, T., R. Engle and J. Wooldridge, 1988, A Capital Asset Pricing Model with Time Varying Covariances, *Journal of Political Economy* 96, 116–131.

[6] Brennan, M. and E. Schwartz, 1980, Analyzing Convertible Bonds, *Journal of Financial and Quantitative Analysis* 15, 907–929.

[7] Campbell, J., 1986, Bond and Stock Returns in a Simple Exchange Model, *Quarterly Journal of Economics* 101, 785–803.

[8] Chan, K.C., G.A. Karolyi, F.A. Longstaff, and A.B. Sanders, 1992, Comparison of Models of the Short-Term Interest Rate, *Journal of Finance* 47, 1209–1227.

[9] Courtadon, G., 1982, The Pricing of Options on Default-Free Bonds, *Journal of Financial and Quantitative Analysis* 17, 75–101.

[10] Cox, J., J. Ingersoll and S. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–407.

[11] Duan, J.-C., 1995, The GARCH Option Pricing Model, *Mathematical Finance* 5, 13-32.

[12] Duan, J.-C., 1996a, Cracking the Smile, *RISK* 9 (December), 55-59.

[13] Duan, J.-C., 1996b, Augmented GARCH(p,q) Process and its Diffusion Limit, to appear in *Journal of Econometrics*.

[14] Duan, J.-C. and K. Jacobs, 1996, A Simple Long-Memory Equilibrium Interest Rate Model, to appear in *Economics Letters*.

[15] Engle, R., D. Lilien and R. Robins, 1987, Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M model, *Econometrica 55*, 391–407.

[16] Engle, R. and V. Ng, 1993, Measuring and Testing the Impact of News on Volatility, *Journal of Finance* 48, 1749– 1778.

[17] Glosten, L., R. Jagannathan and D. Runkle, 1993, Relationship between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance* 48, 1779–1801.

[18] Heynen, R., A. Kemna and T. Vorst, 1994, Analysis of the Term Structure of Implied Volatilities, *Journal of Financial and Quantitative Analysis* 29, 31–56.

[19] Honore, P., 1996, Estimation of a Dynamic One-Factor Continuous-Time Term-Structure Model, unpublished manuscript, Aarhus School of Business.

[20] Hull, J. and A. White, 1987, The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance* 42, 281–299.

[21] Jamshidian, F., 1989, An Exact Bond Option Formula, *Journal of Finance* 44, 205–209.

[22] Lucas, R., 1978, Asset Prices in an Exchange Economy, *Econometrica* 46, 1429–1445.

[23] Mehra, R. and E. Prescott, 1985, The Equity Premium: A Puzzle, *Journal of Monetary Economics* 15, 145–162.

[24] Nelson, D., 1991, Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* 59, 347–370.

[25] Rumsey, J., 1993, The Impact of Assumptions about Taxes on the Estimation of Properties of Interest Rates, University of Toronto working paper.

[26] Vasicek, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.

Figure 1a. The GARCH yield curve is computed by assuming the one-week interest rate equal to its long-run average, 6%. The yield curves corresponding to Vasicek (1977) and Cox, Ingersoll and Ross (1985) are the best possible curves fitted to the GARCH yield curve. All yields are annualized.

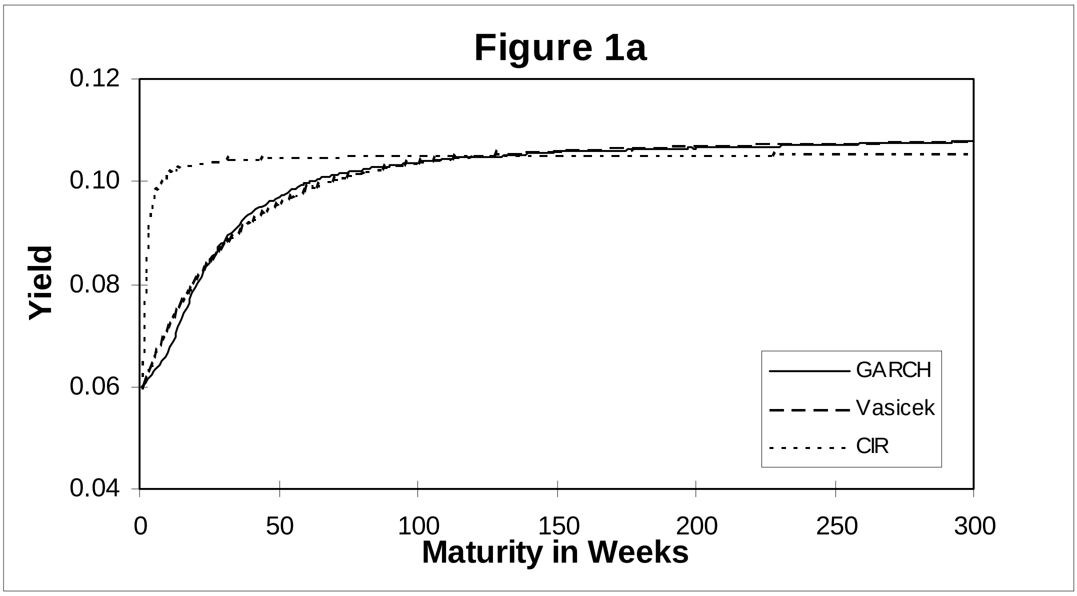

Figure 1b. The yield curves corresponding to Vasicek (1977) and Cox, Ingersoll and Ross (1985) are generated using the parameter values obtained from the fitting in Figure 1a. The one-week rate is set at half of its long-run average, i.e., 3%. All yields are annualized.

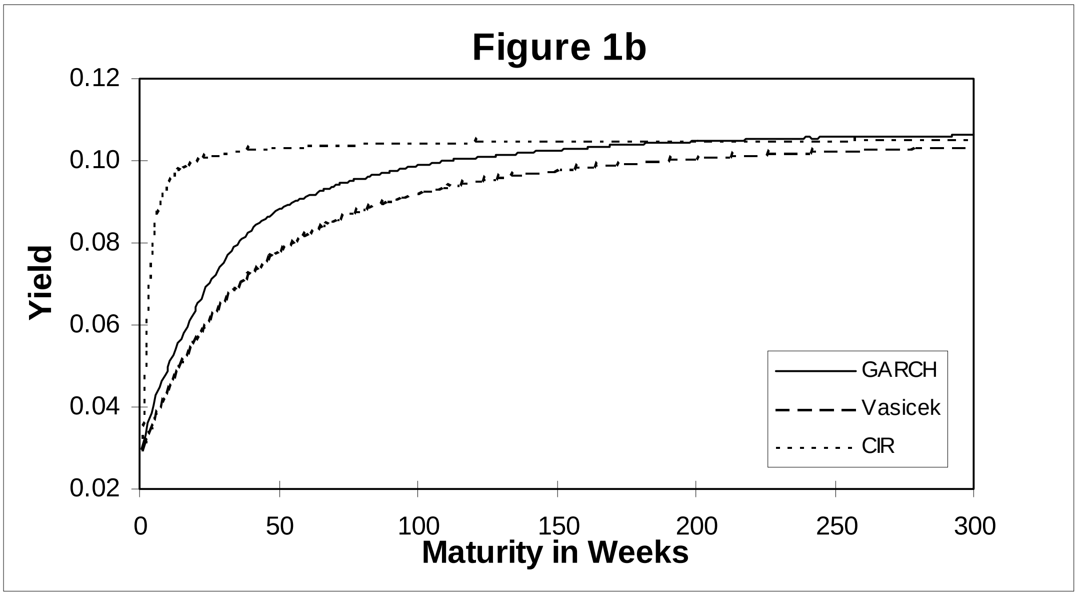

Figure 1c. The yield curves corresponding to Vasicek (1977) and Cox, Ingersoll and Ross (1985) are generated using the parameter values obtained from the fitting in Figure 1a. The one-week rate is set at twice of its long-run average, i.e., 12%. All yields are annualized.

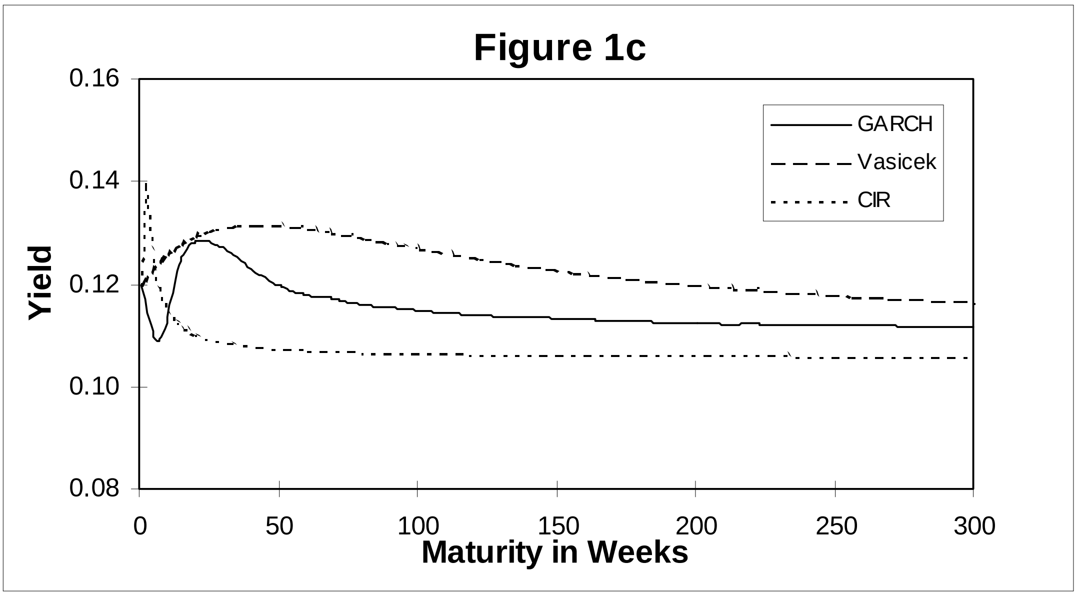

Figure 2a. Options to buy the three-month Treasury bill under three models. The maturity of the option is three months. The one-week interest rate is set equal to its long-run average, 6%. F/X stands for the face value of the underlying Treasury bill over the exercise price. The option prices are stated as per dollar face value of the underlying Treasury bill.

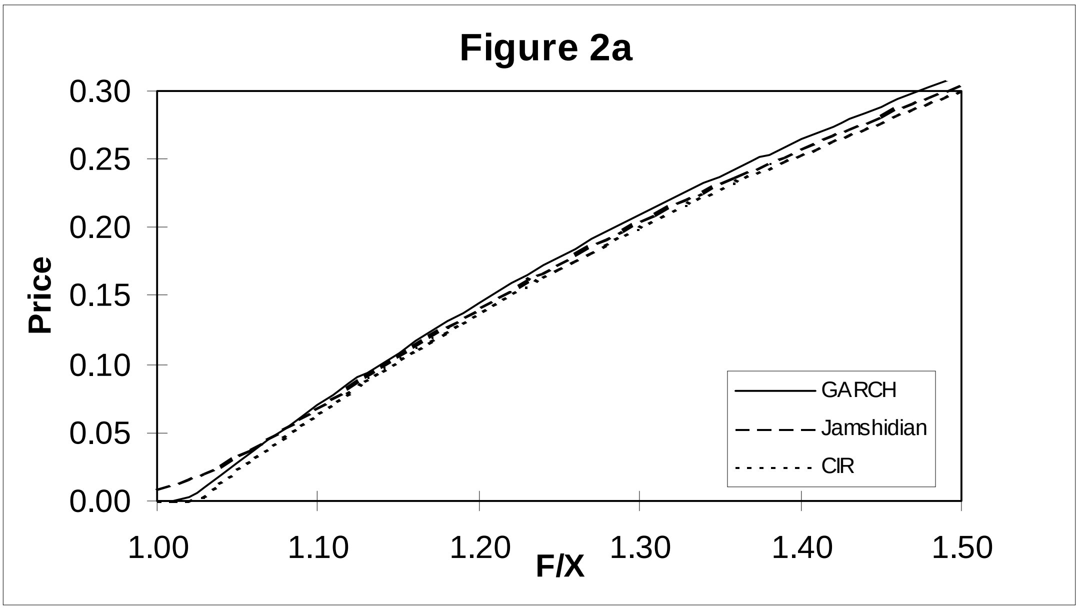

Figure 2b. Options to buy the three-month Treasury bill under three models. The maturity of the option is three months. The one-week interest rate is set at half of its long-run average, i.e., 3%. F/X stands for the face value of the underlying Treasury bill over the exercise price. The option prices are stated as per dollar face value of the underlying Treasury bill.

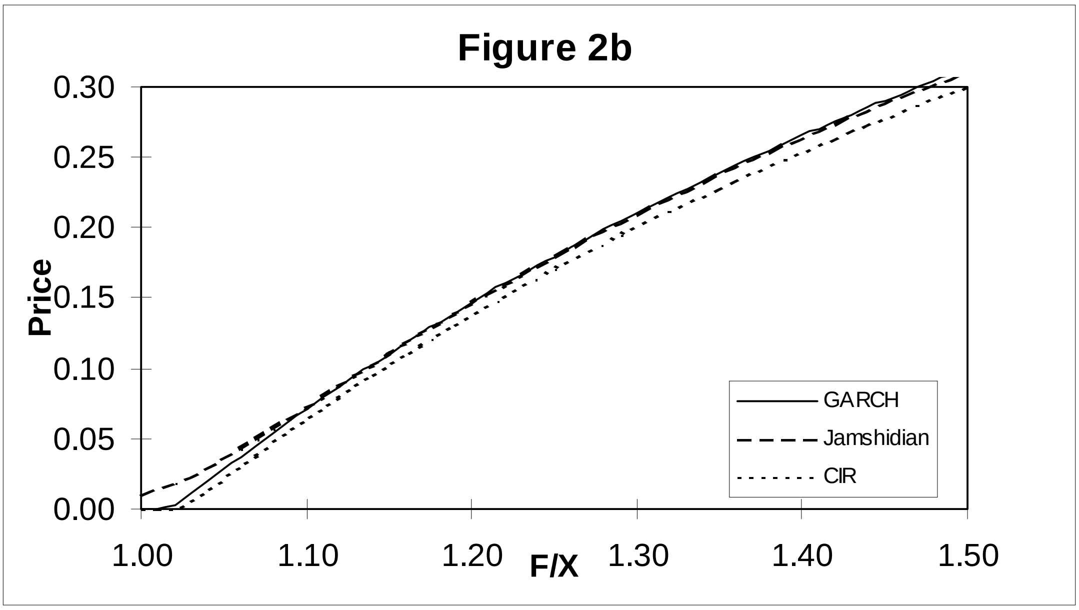

Figure 2c. Options to buy the three-month Treasury bill under three models. The maturity of the option is three months. The one-week interest rate is set at twice of its long-run average, i.e., 12%. F/X stands for the face value of the underlying Treasury bill over the exercise price. The option prices are stated as per dollar face value of the underlying Treasury bill.