A Unified Theory of Option Pricing under Stochastic Volatility – from GARCH to Diffusion

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First Draft: April 1994
Current Draft: October 1996

Abstract

There are two strands of literature on option pricing under stochastic volatility – the bivariate diffusion and GARCH approaches. These two strands of models are unified in this article by a convergence result. The existing bivariate diffusion option pricing models are shown to be the limits of the GARCH option pricing model. Theoretically, the limit result not only unifies these two strands of literature but also provides new insights into the existing bivariate diffusion option pricing models. Operationally, the limit result gives rise to the possibility of interchanging the GARCH and bivariate diffusion models for purposes of estimation and option valuation.

Keywords: Option Pricing, Stochastic Volatility, GARCH Model, Diffusion Process, Local Risk Neutralization, Equilibrium Price Measure, Minimal Martingale Measure.

*The earlier version of this paper was presented in 1995 Western Finance Association conference and 1995 Northern Finance Association conference. The author acknowledges the valuable comments received from the discussants, K. Amin and R. Trevor, and other participants. This research is supported by a grant from the SSHRC of Canada. Correspondence to: J.C. Duan, Department of Finance, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong; Tel: 852-2358 7671; Fax: 852-2358 1749; E-mail: jcduan@ust.hk; Web: http://www.bm.ust.hk/~jcduan.

1 Introduction

There are two strands of literature dealing with the valuation of options when the asset return volatility is stochastic. The first strand approaches option pricing with stochastic volatility in a diffusion framework. Earlier models that allow for stochastic volatility specify the diffusion coefficient as a function of asset price. The compound option pricing model of Geske (1979), the CEV option pricing model of Cox (1975), the displaced diffusion model of Rubinstein (1983) are some examples. These earlier models are of univariate nature. More recently, option pricing with stochastic volatility has been dealt with in a bivariate diffusion framework, in which the volatility of an asset is assumed to follow a separate stochastic process. The examples of this approach are abundant: Hull and White (1987), Scott (1987), Wiggins (1987), Johnson and Shanno (1987), Melino and Turnbull (1990), Stein and Stein (1991) and Heston (1993).

The second strand of literature develops the option pricing model in a GARCH framework. The earlier work by Engle and Mustafa (1992) attempts to extract information from option prices in the GARCH setting. A comprehensive development of the GARCH option pricing theory is first available in Duan (1995). The GARCH option pricing theory has also been extended to pricing currency-related derivative products (Duan and Wei, 1996) and term structure and interest rate derivatives (Duan, 1996a). Kallsen and Taqqu (1994) have also proposed a continuous-time version of the GARCH model, and interestingly they are able to arrive at the GARCH option pricing model of Duan (1995) by an arbitrage-free argument.

The purpose of this article is to unify the GARCH and bivariate diffusion approaches to option pricing under stochastic volatility. The result in turn provides new insights into many existing bivariate diffusion option pricing models. A unified treatment of option pricing with stochastic volatility is made possible by employing the limit result in Duan (1996b). Nelson (1990) is the first one to show that the GARCH process could weakly converge to some bivariate diffusion models. Duan (1996b) shows an enlarged parametric family of GARCH processes, referred to as augmented GARCH process, also converges to a suitable bivariate diffusion limit. This limit process contains most stochastic volatility models appearing in the literature of option pricing. It is Duan's (1996b) generalized limit result that makes the unification of option pricing theory possible.

Our strategy in this article begins by describing the asset price dynamic as an augmented GARCH process. A sequence of economies corresponding to a decreasing length of the basic operational time interval is constructed. For each element of the sequence, an approximate augmented GARCH process is assumed to describe the asset return dynamic in that particular economy. We then invoke the local risk-neutral valuation principle to establish the GARCH option pricing model for this economy. As a result, we obtain two augmented GARCH processes for each economy - one under the data generating probability measure and the other under the locally risk-neutralized probability measure. Shrinking the length of the basic operational time interval to zero yields two limiting bivariate diffusion processes with each governed by its own weak limit of the sequence of probability measures. In addition, we show the weak limit of the locally risk-neutralized probability measures is a minimal martingale measure in the sense of Follmer and Schweizer (1991).

The limit result is used to reexamine several existing option pricing models. In some cases, the limit result strengthens the conclusion of the existing models, whereas in others, we obtain the conditions needed for justifying the existing models. Theoretically, our result offers a solution to the dilemma facing option pricing in the bivariate diffusion framework. In such setup, there are two driving innovations underlying the asset return, but only one underlying asset and the risk-free bond can be used to construct hedge portfolios. This theoretical difficulty is well known in the option literature. The difficulty stems from the fact that the set of equivalent martingale measures is not a singleton. The pricing strategies in this strand of literature are often ad-hoc; for example, assuming volatility is a traded asset, or assuming volatility risk is uncorrelated with aggregate consumption, or assuming the volatility risk premium is a separate constant. The correspondence between the two limiting diffusion processes, derived in this article, therefore provides an alternative option pricing system for the bivariate diffusion setup.

Our limit result also has an important practical significance. It allows for interchanging the GARCH and bivariate diffusion models for purposes of estimation and option valuation. With this limit result, one can easily draw from a vast array of numerical and statistical techniques already developed for the GARCH models and/or diffusion systems.

2 Asset price process from GARCH to diffusion

Consider the nth economy defined over a finite time interval $[0, T]$. Divide this interval into nT subintervals of equal length. Let s denote the length of these subintervals, i.e., $s = 1/n$. Let $\varepsilon_k, k = 1, 2, \cdots$ be a sequence of *i.i.d.* standard normal random variables under the data generating probability measure P of a suitably defined probability space. Define

$$Z_k^{(2)} = |\varepsilon_k - c|^\delta \tag{1}$$

$$Z_k^{(3)} = max(0, c - \varepsilon_k)^\delta \tag{2}$$

$$Z_k^{(4)} = \alpha_4 f[|\varepsilon_k - c|; \delta] + \alpha_5 f[max(0, c - \varepsilon_k); \delta] \tag{3}$$

where $f(z; \delta) = \frac{z^\delta - 1}{\delta}$ for any $z \geq 0$. We define the means of these random variables as

$$\begin{aligned} q_2 &= E^P(Z_k^{(2)}) \\ q_3 &= E^P(Z_k^{(3)}) \\ q_4 &= E^P(Z_k^{(4)}) \end{aligned}$$

where q_2, q_3 and q_4 are three finite constants. The asset price is assumed to follow an approximate augmented GARCH(1,1) process first defined in Duan (1996b). For $k = 1, 2, \cdots, nT$,

$$\ln \frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}} = (r + \omega\sqrt{h_{ks}^{(n)}} - \frac{1}{2}h_{ks}^{(n)})s + \sqrt{h_{ks}^{(n)}}\varepsilon_k\sqrt{s} \tag{4}$$

$$\phi_{(k+1)s}^{(n)} - \phi_{ks}^{(n)} = (\alpha_0 + q_4)s + \phi_{ks}^{(n)}(\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - 1)s +$$

$$\phi_{ks}^{(n)}[\alpha_2(Z_k^{(2)} - q_2) + \alpha_3(Z_k^{(3)} - q_3)]\sqrt{s} + (Z_k^{(4)} - q_4)\sqrt{s} \tag{5}$$

$$h_{ks}^{(n)} = \begin{cases} |\lambda\phi_{ks}^{(n)} - \lambda + 1|^{\frac{1}{\lambda}}, & \text{if } \lambda > 0 \\ \exp(\phi_{ks}^{(n)} - 1), & \text{if } \lambda = 0 \end{cases} \tag{6}$$

The variable s is the length of the approximating time interval. The specific specification for the conditional mean allow us to interpret the parameter ω as the unit risk premium of the asset return. For analytical convenience, we assume that the interest rate r and the unit risk premium ω are constant. If these two parameters are stochastic but predictable, then the results derived in this article will remain intact. Additional structures are, however, needed in order to derive the dynamic for the interest rate and for the unit risk premium.[1]

If the length of time interval is one, i.e., $s = 1$, different parameter restrictions yield many existing GARCH models. If $\lambda = 1, c = 0, \delta = 2, \alpha_3 = 0, \alpha_4 = 0$ and $\alpha_5 = 0$, the augmented GARCH model reduces to a linear GARCH-Mean model. If $\lambda = 0, c = 0, \delta = 1, \alpha_2 = 0$ and $\alpha_3 = 0$, then the augmented GARCH model specializes to the exponential GARCH-Mean process. Other versions of the GARCH model can also be produced by setting appropriate parameter values (see Duan, 1996b).

Define σ_{ij} as element (i, j) of $Var^P(\varepsilon_k, Z_k^{(2)}, Z_k^{(3)}, Z_k^{(4)})$, a constant matrix. Let P_n denote the distributions of the augmented GARCH(1,1) process in the nth economy. In following lemma, we state a simplified version of Theorem 3 of Duan (1996b).

Lemma 1. The augmented GARCH(1,1) model defined in (4), (5) and (6) weakly converges to the following bivariate diffusion model:

$$d\ln X_t = (r + \omega\sqrt{h_t} - \frac{1}{2}h_t)dt + \sqrt{h_t}dW_{1t} \tag{7}$$

$$d\phi_t = [\alpha_0 + q_4 + (\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - 1)\phi_t]dt + v_t\rho_t dW_{1t} + v_t\sqrt{1-\rho_t^2}dW_{2t} \tag{8}$$

$$h_t = \begin{cases} |\lambda\phi_t - \lambda + 1|^{\frac{1}{\lambda}}, & \text{if } \lambda > 0 \\ \exp(\phi_t - 1), & \text{if } \lambda = 0 \end{cases} \tag{9}$$

where

$$\begin{aligned} v_t &= \sqrt{\sigma_4^2 + 2(\alpha_2\sigma_{24} + \alpha_3\sigma_{34})\phi_t + (\alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + 2\alpha_2\alpha_3\sigma_{23})\phi_t^2)}, \\ \rho_t &= v_t^{-1}[\sigma_{14} + \phi_t(\alpha_2\sigma_{12} + \alpha_3\sigma_{13})], \text{ and} \\ & W_{1t} \text{ and } W_{2t} \text{ are two independent } P^*\text{-Wiener processes and} \\ & P^* \text{ denotes the weak limit of } P_n. \end{aligned}$$

Proof: see Theorem 3 of Duan (1996b).

[1] The assumption of constant risk-free rate and unit risk premium is not theoretically trivial. Consider an economy with one risky and one risk-free asset, in which the representative agent has a time-additive, separable expected utility function. If the risky asset, i.e., the market portfolio, has a stochastic volatility in this economy, then it implies that the risk-free interest rate and unit risk premium are stochastic. Strictly speaking, the assumption of constant interest rate and risk premium precludes the market portfolio, although not individual assets, from having a stochastic volatility. If one wants to model the market portfolio with stochastic volatility, letting the interest rate and risk premium be stochastic is essential in keeping an overall consistent option pricing model.

The limit model is well defined because $|\rho_t| \leq 1$ and v_t is bounded away from zero. The last two terms involved two independent Wiener processes in the equation for ϕ_t can be, if one so chooses, combined into $v_t dZ_t$ where Z_t is a newly-defined Wiener process with ρ_t being the correlation between W_{1t} and Z_t.

This limit process specializes to many existing bivariate diffusion models commonly used in the literature. If $\lambda = 0, c = 0, \delta = 1, \alpha_2 = 0$ and $\alpha_3 = 0$, the stochastic variance has the following dynamic:

$$\begin{aligned} d\ln h_t &= [\alpha_0 + \frac{2}{\sqrt{2\pi}}(\alpha_4 + \frac{\alpha_5}{2}) - \alpha_4 - \alpha_5 + \alpha_1 - 1 + (\alpha_1 - 1)\ln h_t]dt - \\ & \frac{\alpha_5}{2} dW_{1t} + |\alpha_4 + \frac{\alpha_5}{2}|\sqrt{\frac{\pi-2}{\pi}} dW_{2t}. \end{aligned} \tag{10}$$

The model corresponding to the above system is the exponential GARCH model of Nelson (1991). This diffusion specification for stochastic variance is used in Wiggins (1987) for pricing options with stochastic volatility.

If a non-linear asymmetric GARCH model of Engle and Ng (1993) is adopted, i.e., $\lambda = 1, \delta = 2, \alpha_3 = 0, \alpha_4 = 0$ and $\alpha_5 = 0$, then

$$dh_t = \{\alpha_0 + [\alpha_1 + \alpha_2(1 + c^2) - 1]h_t\}dt - 2c\alpha_2 h_t dW_{1t} + \sqrt{2}\alpha_2 h_t dW_{2t}. \tag{11}$$

The limit of the GARCH model by Glosten, et al (1993) shares the same diffusion limit with the non-linear asymmetric GARCH model. If we further restrict the model by setting $\alpha_0 = 0$, it gives rise to the specification of Hull and White (1987). When the asymmetry parameter c is set to zero, the approximating model is known as the linear GARCH(1,1) process in Bollerslev (1986) and Taylor (1986). Under this model, the return and volatility innovations are uncorrelated in the nth economy and become independent in the limit.

The stochastic volatility model of Scott (1987), Stein and Stein (1991) and Heston (1993) can also be obtained by setting $\lambda = \frac{1}{2}, c = 0, \delta = 1, \alpha_2 = 0$ and $\alpha_3 = 0$. In such case,

$$d\sqrt{h_t} = sign(\phi_t^*)d\phi_t^* + dL_t^0(\phi_t^*) \tag{12}$$

$$d\phi_t^* = [\frac{\alpha_0 + q_4 - \alpha_1 + 1}{2} + (\alpha_1 - 1)\phi_t^*]dt + \frac{\alpha_5}{4}dW_{1t} + |\frac{2\alpha_4 + \alpha_5}{4}|\sqrt{\frac{\pi - 2}{\pi}}dW_{2t} \tag{13}$$

$$q_4 = \frac{2\alpha_4 + \alpha_5}{\sqrt{2\pi}} - \alpha_4 - \alpha_5 \tag{14}$$

where $\phi_t^* = \frac{\phi_t + 1}{2}$, and $L_t^0(\phi_t^*)$ denotes the local time of ϕ_t^* at 0. In Scott (1987), Stein and Stein (1991) and Heston (1993), ϕ_t^* is interpreted as the conditional standard deviation. The model above, however, follows the interpretation given in Duan (1996b). Duan argues that viewing ϕ_t^* as the standard deviation is inappropriate because ϕ_t^* can take on negative values. Use of the correct interpretation will not change the basic option pricing result in Scott (1987), Stein and Stein (1991) and Heston (1993), because h_t and ϕ_t^{*2} share the same dynamic.

3 Equilibrium pricing measure and local risk neutralization

We first define an equilibrium pricing measures by

$$dQ = e^{(r-\beta)T} \frac{U'(C_T)}{U'(C_0)} dP \tag{15}$$

where $U(C_t)$ denotes the utility function of consumption C_t used in the standard intertemporal time-additive and separable model of Lucas (1978), and β is the subjective intertemporal discount rate.

Following Duan (1995), we define the concept of local risk-neutralization which plays the key role in our derivation of the option pricing theory under stochastic volatility. Let $\mathcal{F}_{ks}^{(n)}$ be the information set available up to time ks in the nth economy.

Definition. The equilibrium pricing measure Q is said to satisfy the local risk-neutral valuation relationship (LRNVR) in the nth economy if

1. Q is mutually absolutely continuous with respect to P,
2. $ln\frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}}|\mathcal{F}_{(k-1)s}^{(n)}$ has a Q-normal distribution with $E^Q(\frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}}|\mathcal{F}_{(k-1)s}^{(n)}) = e^{rs}$ and $Var^Q(ln\frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}}|\mathcal{F}_{(k-1)s}^{(n)}) = Var^P(ln\frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}}|\mathcal{F}_{(k-1)s}^{(n)})$ almost surely with respect to P.

The LRNVR is shown by Duan (1995) to be valid in the GARCH framework when, for example, the utility function is of constant relative risk aversion and changes in the logarithmic aggregate consumption are distributed normally with constant mean and variance under measure P. The constant mean and variance assumption is simply to ensure that the one-period interest rate remains constant in equilibrium. The LRNVR still holds when the mean and variance are stochastic, but the interest rate can no longer be a constant.

The LRNVR is automatically satisfied in the bivariate diffusion model, and can be proved by applying a martingale representation to the equilibrium pricing measure and then invoking Girsanov's theorem. The LRNVR is, however, insufficient in a bivariate diffusion model for obtaining a unique option price. In the complete market setting of Harrison and Kreps (1979) and Harrison and Pliska (1981), the set of equivalent martingale measures is a singleton, whereas in the stochastic volatility model, the set of equivalent martingale measures contains infinite number of elements with all of them satisfying the LRNVR. The key to our approach is restrict the attention to a sequence of pairs of distributions (data generating and locally risk-neutralized) with each pair corresponding to the asset return dynamic of the nth economy.

We use in this article the LRNVR to obtain a characterization of the asset return dynamic under the equilibrium pricing measure Q. Although the resulting model implicitly depends on preferences, it is only indirectly through the unit risk premium of the underlying asset. Interestingly, Kallsen and Taqqu (1994) show that the GARCH option pricing model of Duan (1995) can also be established by an arbitrage-free argument. Their result therefore implies that an arbitrage-free option pricing model is not necessarily preference-free. From the standpoint of implementation, the unit risk premium is, either explicitly or implicitly, part of an empirical specification for the asset return. All parameters required for option pricing under the GARCH specification are therefore identified directly under the data generating measure P.

Theorem 1. Under the equilibrium pricing measure Q that satisfies the LRNVR in the nth economy, the augmented GARCH process has the following dynamic:

$$\ln \frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}} = (r - \frac{1}{2}h_{ks}^{(n)})s + \sqrt{h_{ks}^{(n)}}\varepsilon_k^* \sqrt{s} \tag{16}$$

$$\begin{aligned}\phi_{(k+1)s}^{(n)} - \phi_{ks}^{(n)} &= (\alpha_0 + q_4)s + \phi_{ks}^{(n)}(\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - 1)s + (Z_k^{*(4)}(s) - q_4)\sqrt{s} + \\ &\quad \phi_{ks}^{(n)}[\alpha_2(Z_k^{*(2)}(s) - q_2) + \alpha_3(Z_k^{*(3)}(s) - q_3)]\sqrt{s}\end{aligned} \tag{17}$$

$$h_{ks}^{(n)} = \begin{cases} |\lambda\phi_{ks}^{(n)} - \lambda + 1|^{\frac{1}{\lambda}}, & \text{if } \lambda > 0 \\ \exp(\phi_{ks}^{(n)} - 1), & \text{if } \lambda = 0 \end{cases} \tag{18}$$

where

$$\begin{aligned}\varepsilon_k^* &= \varepsilon_k + \omega\sqrt{s} \ \text{ with } \ \varepsilon_k^* \sim N(0,1), \\ Z_k^{*(2)}(s) &= |\varepsilon_k^* - c - \omega\sqrt{s}|^{\delta}, \\ Z_k^{*(3)}(s) &= max(0, c + \omega\sqrt{s} - \varepsilon_k^*)^{\delta}, \\ Z_k^{*(4)}(s) &= a_4 f[|\varepsilon_k^* - c - \omega\sqrt{s}|; \delta] + a_5 f[max(0, c + \omega\sqrt{s} - \varepsilon_k^*); \delta].\end{aligned}$$

Proof: see Appendix.

Letting $s = 1$, Theorem 1 leads to the augmented GARCH option pricing model, a generalization of Duan (1995). In accordance with Theorem 1, local risk-neutralization results in a mean shift for the asset return. The essential distributional characteristics of the asset return, however, remain unchanged. Local risk-neutralization also results in a corresponding shift in the innovations that govern the auxiliary variable ϕ_t. Since the conditional variance is predictable, this shift does not have any effect locally, i.e., for the next period. It nevertheless affects the global behavior of the conditional variance process. Failure to retain the property of global risk neutralization does not adversely affect one's ability to derive an operational option pricing model. This is because the shift in innovations is based on the unit risk premium parameter ω that can always be estimated along with the parameters governing the volatility process.

Let Q_n denote the distributions generated by the model in Theorem 1. An important limit result is now in order.

Theorem 2. The locally risk-neutralized augmented GARCH(1,1) process in Theorem 1 weakly converges to the following diffusion system:

$$d\ln X_t = (r - \frac{1}{2}h_t)dt + \sqrt{h_t}dW_{1t}^* \tag{19}$$

$$
\begin{aligned}
d\phi_t &= [\alpha_0 + q_4 - \eta_0 + (\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - \eta_1 - 1)\phi_t]dt + \\
&\quad v_t \rho_t dW_{1t}^* + v_t \sqrt{1-\rho_t^2} dW_{2t}^* \qquad (20) \\
h_t &= \begin{cases} |\lambda\phi_t - \lambda + 1|^{\frac{1}{\lambda}}, & \text{if } \lambda > 0 \\ \exp(\phi_t - 1), & \text{if } \lambda = 0 \end{cases} \qquad (21)
\end{aligned}
$$

where

$$
\begin{aligned}
v_t &= \sqrt{\sigma_4^2 + 2(\alpha_2\sigma_{24} + \alpha_3\sigma_{34})\phi_t + (\alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + 2\alpha_2\alpha_3\sigma_{23})\phi_t^2)}, \\
\rho_t &= v_t^{-1}[\sigma_{14} + \phi_t(\alpha_2\sigma_{12} + \alpha_3\sigma_{13})], \\
\eta_0 &= \omega\sigma_{14}, \quad \eta_1 = \omega(\alpha_2\sigma_{12} + \alpha_3\sigma_{13}), \\
W_{1t}^* &= W_{1t} + \omega t \ \text{ and } \ W_{2t}^* = W_{2t} \ \text{ are two independent } Q^*\text{-Wiener processes,} \\
&\quad Q^* \text{denotes the weak limit of measure } Q_n.
\end{aligned}
$$

Proof: see Appendix.

In terms of (19), the result is a standard one. The contribution of Theorem 2 is the dynamic presented in (20). This theorem implies that the auxiliary process must undergo a mean shift when a change in measures takes place. The adjustment terms are captured by two parameters η_0 and η_1, which are in turn determined by the correlation structure, with respect to the data generating probability measure P, between the innovation for the asset return and those for the auxiliary variable ϕ_t.

4 Minimal martingale measure and option pricing

Follmer and Schweizer (1991) introduce the concept of minimal martingale measure which has been used in the literature as a way of dealing with too many equivalent martingale measures. Schweizer (1992) and Colwell and Elliott (1993) are two examples. An intuitive justification for using the minimal martingale measure is that the risk orthogonal to the asset return should not be compensated, i.e., zero risk premium for orthogonal risk. Although this idea seems intuitive when applied to the market portfolio, it is not clear whether one can assign a zero premium to the risk orthogonal to an individual asset. In this section, we establish that for pricing contingent claims, the weak limit of the locally risk-neutralized probability measures is a minimal martingale measure. This result provides an exceedingly simple way for obtaining the option pricing result for a given bivariate diffusion model.

Let $\mathcal{F}_t$ be the information filtration generated by W_{1t} and W_{2t} up to time t, and $\mathcal{F}(X_t)$ the one generated by X_t up to time t. Clearly, $\mathcal{F}(X_t)$ is contained in $\mathcal{F}_t$, and the limit induces an expansion in the filtration. Any probability measure π is referred to as an equivalent martingale measure if π and P^* share the null sets and the discounted asset price, i.e., $e^{-rt}X_t$, is a $(\pi, \mathcal{F}_t)$-martingale. The minimal martingale is an equivalent martingale measure that for any local $(P^*, \mathcal{F}_t)$-martingale that is orthogonal to $\mathcal{F}(X_t)$ remains a local $(\pi, \mathcal{F}_t)$-martingale. The following corollary can greatly simplify the derivation of the option pricing result for any member of the bivariate diffusion family described in Lemma 1. This corollary immediately follows from Theorem 2 because $W_{1t}^* = W_{1t} + \omega t$ and $W_{2t}^* = W_{2t}$.

Corollary 1. The weak limit of Q_n, i.e., Q^*, is a minimum martingale measure.

The minimal martingale measure can be used to price contingent claims such as options on X_t because they are $\mathcal{F}(X_t)$-measurable claims. The result in Corollary 1 is, in some sense, peculiar and thus calls for further elaboration. Any $\mathcal{F}_t$-measurable contingent claim, even though it is not $\mathcal{F}(X_t)$-measurable, will be assigned a unique price by Q^*. This conclusion results from the expansion of filtration from $\mathcal{F}(X_t)$ to $\mathcal{F}_t$ in the limit. As proved in the preceding section, the locally risk-neutralized measure Q_n in the nth discrete-time economy is only applicable to $\mathcal{F}(X_{ks}^{(n)})$-measurable contingent claims, such as options on $X_{ks}^{(n)}$. The limit result has enlarged the filtration, which is not needed for option pricing but nevertheless yields a model price for the general contingent claim. This peculiar phenomenon arises from using the GARCH model, a univariate non-Markovian discrete-time process, to approximate a bivariate diffusion model. The risk orthogonal to the asset return innovation in the limit economy must originate from the uncorrelated component of some transformation of the asset return (see equations (1), (2) and (3)). Under the equilibrium pricing measure Q, this orthogonal risk component continues to be orthogonal to the asset return innovation in nth discrete-time economy, because the correlation structure remains unchanged after a change of measures if it satisfies the LRNVR. It is therefore not surprisingly that this form of orthogonal risk becomes independent of the asset return innovation in the limit economy and is pricing-irrelevant.

Our result thus poses no conceptual difficulty when is used to reconcile the existing bivariate diffusion option pricing literature. If the risk orthogonal to the underlying asset return constitutes part of the return on another primary asset, e.g., volatility is a traded asset, the model immediately becomes a straightforward complete market economy in which the well-known contingent claim pricing result prevails. If the orthogonal risk is not part of the return on another primary asset, the result in this article provides a solution to the option pricing problem. Since any two bivariate diffusion models with only a difference in their Brownian innovations have the same distribution (weakly equivalent), it is futile to distinguish two models unless a specific economic factor can be further attached to the volatility variable. Until then, the minimum martingale measure appears to be the most sensible result. It is also clear that the pricing problem of the latter case is a more relevant economic issue, and its solution is provided in this article.

5 Bivariate diffusion option pricing models revisited

As discussed in Section 2, many bivariate diffusion models in the literature are special cases of the limit model. Theorem 2 can thus be directly applied to these models to derive their corresponding option pricing results. The minimal martingale measure result in Corollary 1 is, however, a more convenient device in deriving option pricing models. In this section, we use this approach to analyze the option pricing models of Hull and White (1987), Wiggins (1987), Scott (1987), Stein and Stein (1991) and Heston (1993).

Hull and White's (1987) bivariate diffusion model under P^*can be expressed as

$$\begin{aligned} dX_t &= (r + \omega\sqrt{h_t})X_t dt + \sqrt{h_t} X_t dW_{1t} && (22) \\ dh_t &= (\beta_0 + \beta_1 h_t)dt + \psi_1 h_t dW_{1t} + \psi_2 h_t dW_{2t} && (23) \end{aligned}$$

which is a re-parameterization of (7) and (11). Note that we have slightly generalized the model by introducing a parameter β_0, which equals zero in Hull and White's (1987) model. By Corollary 1, the system under Q^* must become

$$\begin{aligned} dX_t &= r X_t dt + \sqrt{h_t} X_t dW^*_{1t} && (24) \\ dh_t &= [\beta_0 + (\beta_1 - \psi_1\omega)h_t]dt + \psi_1 h_t dW^*_{1t} + \psi_2 h_t dW^*_{2t} && (25) \end{aligned}$$

When $\psi_1 = 0$, the innovations of the return and volatility are independent, and the premium for volatility risk equals zero. This justifies the first result of Hull and White (1987). As to the correlated case of Hull and White, our result implies that the volatility risk premium can become zero only when the unit risk premium for the underlying asset, ω, is zero. In other words, unless both ψ_1 and ω are zero, the volatility risk must be priced.

For Wiggins' (1987) model, we re-parameterize the system in (7) and (10) and state them as follows:

$$\begin{aligned} dX_t &= (r + \omega\sqrt{h_t})X_t dt + \sqrt{h_t} X_t dW_{1t} && (26) \\ dlnh_t &= (\beta_0 + \beta_1 lnh_t)dt + \psi_1 dW_{1t} + \psi_2 dW_{2t} && (27) \end{aligned}$$

Invoking Corollary 1 yields the system under measure Q^* as follows:

$$\begin{aligned} dX_t &= r X_t dt + \sqrt{h_t} X_t dW^*_{1t} && (28) \\ dlnh_t &= (\beta_0 - \psi_1\omega + \beta_1 lnh_t)dt + \psi_1 dW^*_{1t} + \psi_2 dW^*_{2t} && (29) \end{aligned}$$

In contrast to Wiggins' (1987) conclusion, the result above suggests that the volatility risk orthogonal to the asset return is not compensated irrespective of whether the underlying asset is a market portfolio or individual security. In terms of Wiggins's (1987) notation, our result implies $\phi(\cdot) = 0$. The discounted expected option payoff, with respect to the dynamic in (28) and (29), must satisfy the Kolmogorov backward equation, which in turn leads to Wiggins' (1987) partial differential equation for option price. One should not over-extend the conclusion of the above result, however. As discussed in the preceding section, the orthogonal risk in the context of stochastic volatility model is not equivalent to the orthogonal risk in general, for example, another asset with an independent return. It nevertheless suggests that a hedge portfolio consisting of the underlying asset and its option should command a zero risk premium if the hedge portfolio's return is uncorrelated with the underlying asset.

The model of Scott (1987), Stein and Stein (1991) and Heston (1993), which is stated earlier in (7), (12), (13) and (14), can be re-parameterized to yield

$$\begin{aligned} dX_t &= (r + \omega\sqrt{h_t})X_t dt + \sqrt{h_t} X_t dW_{1t} && (30) \\ d\phi^*_t &= (\beta_0 + \beta_1 \phi^*_t)dt + \psi_1 dW_{1t} + \psi_2 dW_{2t} && (31) \\ h_t &= \phi^{*2}_t && (32) \end{aligned}$$

Applying Corollary 1 gives rise to the system under measure Q^*:

$$dX_t = rX_t dt + \sqrt{h_t} X_t dW_{1t}^* \tag{33}$$

$$d\phi_t^* = (\beta_0 - \psi_1 \omega + \beta_1 \phi_t^*)dt + \psi_1 dW_{1t}^* + \psi_2 dW_{2t}^* \tag{34}$$

$$h_t = \phi_t^{*2} \tag{35}$$

The price of an option can be computed by taking expectation of its payoff using the above system and then discounting it by the risk-free rate. The partial differential equation of Scott (1987), Stein and Stein (1991) and Heston (1993) is satisfied by applying the Kolmogorov backward equation to this entity. There is nevertheless one important difference between their results and the one in (33), (34) and (35). The volatility risk premium is left unspecified in their models, whereas our result implies a definitive relationship between the risk premium for the underlying asset and that for the volatility. Similar to the earlier discussion on Hull and White's (1987) model, the volatility risk premium can become zero only when $\psi_1 = 0$, or $\omega = 0$, or both.

The limit result also suggests a way of approaching the model of Johnson and Shanno (1987) and that of Melino and Turnbull (1990). As pointed out in Duan (1996b), the limit of the augmented GARCH process can be made to include their models by defining h_t as a product of the auxiliary variable ϕ_t and a power function of X_t. There is one inherent technical difficulty, however. Since power function generally fails to satisfy the Lipschitz and growth conditions either at infinity or zero, the existence and uniqueness of the solution to such bivariate diffusion model become unknown. If one assumes that the solution to their bivariate diffusion models exists and is unique, then the weak limit of the locally risk-neutralized measures will continue to be a minimal martingale measure.

6 Conclusion

The literature on option pricing under stochastic volatility can be grouped into two categories – the bivariate diffusion and GARCH approaches. These two strands of option pricing models are unified by a convergence result presented in this article. The GARCH option pricing model is shown to weakly converge to a bivariate diffusion option pricing model, and this model can be specialized to many existing option pricing models. The convergence result for option pricing is built upon the idea of Nelson (1990) and Duan (1996b) that the GARCH process can be used to approximate the bivariate diffusion model. Our result goes beyond providing new theoretical insights into the literature of option pricing with stochastic volatility. The convergence result offers a practical way of implementing the bivariate diffusion option pricing models. In the bivariate diffusion framework, the volatility is not observable and parameter estimation can be difficult. The GARCH model, on the other hand, can be easily estimated and the conditional volatility is readily available. This fact makes the use of the GARCH approach attractive for both estimation and option pricing, even if one prefers to model the asset return as a bivariate diffusion process. The limit result can also be used in a reverse manner. One can tap into the set of numerical option pricing techniques already developed for diffusion systems to compute option prices even if the GARCH option pricing model is one's preferred choice.

7 Appendix

Proof of Theorem 1.

Invoking the LRNVR, we have under the equilibrium pricing measure Q

$$ln\frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}} = (r - \frac{1}{2}h_{ks}^{(n)})s + \sqrt{h_{ks}^{(n)}}\varepsilon_{ks}^{*}\sqrt{s}$$

where $\varepsilon_{ks}^{*}|\mathcal{F}_{(k-1)s}^{(n)} \sim N(0,1)$. This relationship can also be written as

$$ln\frac{X_{ks}^{(n)}}{X_{(k-1)s}^{(n)}} = (r + \omega\sqrt{h_{ks}^{(n)}} - \frac{1}{2}h_{ks}^{(n)})s + \sqrt{h_{ks}^{(n)}}(\varepsilon_{ks}^{*} - \omega\sqrt{s})\sqrt{s}$$

Comparing it to (4) yields $\varepsilon_{ks}^{*} = \varepsilon_{ks} + \omega\sqrt{s}$. Substituting this relation into (5) gives rise to the desirable result.□

Proof of Theorem 2.

The approximating augmented GARCH(1,1) process can be written in a difference form

$$\begin{aligned}
\Delta lnX_{ks}^{(n)} &= (r - \frac{1}{2}h_{ks}^{(n)})s + \sqrt{h_{ks}^{(n)}}\varepsilon_{ks}^{*}\sqrt{s} \\
\Delta\phi_{(k+1)s}^{(n)} &= (\alpha_0 + q_4)s + \phi_{ks}^{(n)}(\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - 1)s + \\
&\quad \phi_{ks}^{(n)}[\alpha_2(Z_k^{*(2)}(s) - q_2(s)) + \alpha_3(Z_k^{*(3)}(s) - q_3(s))]\sqrt{s} + \\
&\quad (Z_k^{*(4)}(s) - q_4(s))\sqrt{s} + (q_4(s) - q_4)\sqrt{s} + \\
&\quad \phi_{ks}^{(n)}[\alpha_2(q_2(s) - q_2) + \alpha_3(q_3(s) - q_3)]\sqrt{s}.
\end{aligned}$$

where

$$\begin{aligned}
q_2(s) &= E^Q(Z_k^{*(2)}(s)), \\
q_3(s) &= E^Q(Z_k^{*(3)}(s)), \\
q_4(s) &= E^Q(Z_k^{*(4)}(s)).
\end{aligned}$$

Suppose

$$\begin{aligned}
q_4(s) - q_4 &= -\eta_0\sqrt{s} + O(s), \\
\alpha_2(q_2(s) - q_2) + \alpha_3(q_3(s) - q_3) &= -\eta_1\sqrt{s} + O(s).
\end{aligned}$$

where η_0 and η_1 are two finite constants. The system for $\phi_{ks}^{(n)}$ can be written as

$$\begin{aligned}
\Delta\phi_{(k+1)s}^{(n)} &= (\alpha_0 + q_4 - \eta_0)s + (\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - \eta_1 - 1)\phi_{ks}^{(n)}s + \\
&\quad \phi_{ks}^{(n)}[\alpha_2(Z_k^{*(2)}(s) - q_2(s)) + \alpha_3(Z_k^{*(3)}(s) - q_3(s))]\sqrt{s} + (Z_k^{*(4)}(s) - q_4(s))\sqrt{s} + \\
&\quad O(s^{\frac{3}{2}})
\end{aligned}$$

The term $O(s^{\frac{3}{2}})$ can be ignored in the limit because its order is higher than s. Invoking Lemma 1 again yields

$$\begin{aligned} d\ln X_t &= (r - \frac{1}{2}h_t)dt + \sqrt{h_t}dW_{1t}^*, \quad \text{and} \\ d\phi_t &= [\alpha_0 + q_4 - \eta_0 + (\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - \eta_1 - 1)\phi_t]dt + v_t\rho_t dW_{1t}^* + v_t\sqrt{1-\rho_t^2}dW_{2t}^* \end{aligned}$$

where W_{1t}^* and W_{2t}^* are two independent Q^*-Wiener processes. Note that σ_{ij} is the same as the one under measure P because $Z_k^{*(2)}(s)$ converges to $Z_k^{(2)}$ in probability, and the same is true for $Z_k^{*(3)}(s)$ and $Z_k^{*(4)}(s)$. It remains to show that η_0, η_1, W_{1t}^* and W_{2t}^* are as stated in the theorem.

Consider any differentiable function of $g(\varepsilon)$ such that if ε approaches either $+\infty$ or $-\infty$, then $g(\varepsilon)e^{-\frac{1}{2}\varepsilon^2}$ goes to 0. It follows from Rubinstein (1976, p.421) that if ε is a standard normal random variable under a probability measure π, then

$$E^\pi[g'(\varepsilon)] = Cov^\pi[\varepsilon, g(\varepsilon)].$$

This relationship is used to derive the following result for $i = 2, 3$ or 4. For the following application, function g_i is not differentiable only at one point. Since this point has a zero probability measure, we can still apply the above result. Thus,

$$\begin{aligned} q_i(s) - q_i &= E^Q[g_i(\varepsilon_k^* - \omega\sqrt{s})] - E^P[g_i(\varepsilon_k)] \\ &= E^P[g_i(\varepsilon_k - \omega\sqrt{s})] - E^P[g_i(\varepsilon_k)] \\ &= -E^P[g_i'(\varepsilon_k)]\omega\sqrt{s} + O(s) \\ &= -Cov^P[\varepsilon_k, g_i(\varepsilon_k)]\omega\sqrt{s} + O(s) \\ &= -\omega\sigma_{1i}\sqrt{s} + O(s) \end{aligned}$$

Thus,

$$\begin{aligned} \eta_0 &= \omega\sigma_{14}, \\ \eta_1 &= \omega(\alpha_2\sigma_{12} + \alpha_3\sigma_{13}). \end{aligned}$$

This in turn allows us to rewrite the limit model as follows:

$$\begin{aligned} dlnX_t &= (r + \omega\sqrt{h_t} - \frac{1}{2}h_t)dt + \sqrt{h_t}d(W_{1t}^* - \omega t), \text{ and} \\ d\phi_t &= [\alpha_0 + q_4 + (\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - 1)\phi_t]dt + v_t\rho_t d(W_{1t}^* - \omega t) + v_t\sqrt{1-\rho_t^2}dW_{2t}^* \end{aligned}$$

By comparing this limit model with the one in Lemma 1, the results for W_{1t}^* and W_{2t}^* are established.□

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